A Crisp Representation for Fuzzy $\mathcal{SHOIN}$ with Fuzzy Nominals and General Concept Inclusions

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Agenda

Preliminaries

- Ontologies are a core element in the layered architecture of the Semantic Web
- Description Logics (DLs) are a family of logics for representing structured knowledge
- DLs have been proved to be very useful as ontology languages
- Standard for ontology representation: OWL Web Ontology Language:
  - OWL Lite
  - OWL DL, the most used, nearly equivalent to $\mathcal{SHOIN}(\mathcal{D})$ DL
  - OWL Full, undecidable

Motivation

- Classical ontologies/DLs are not appropriate for uncertain or imprecise knowledge
- Examples:
  - German is generally spoken in Germany, Austria and Switzerland
  - An inn is a cheap and small hotel
- Vagueness is inherent to a lot of real-world application domains
  - The Semantic Web will not be fully operative as long as it does not provide means to manage it
- One solution: Fuzzy DLs (DLs extended with fuzzy sets theory)

Motivation (2)

- OWL may be extended to a fuzzy DL-based language e.g. FuzzyOWL
  - The large number of resources available should be adapted
  - In particular, we need reasoners
- Reasoning within expressive DLs has a very high worst-case complexity
  - Significant gap between the design of a decision procedure and the achievement of a practical implementation
  - The experience with crisp DLs induces us to think that developing highly optimized implementations will be a hard task where ad-hoc mechanisms should be used for every particular fuzzy DL
  - In fact, there is no implemented reasoner for $f\mathcal{SHOIN}$
  - Nothing is known about the efficiency of existing reasoners:
    - fuzzyDL (f_{KD}[1]$\mathcal{SHIF}(\mathcal{D})$, $f_{Ł}\mathcal{SHIF}(\mathcal{D})$)
    - Fire ($f_{KD}\mathcal{SHIN}$)

[1] subscript stands for the implication used, which specifies the other operators

Motivation (3)

- Alternative way to obtain fuzzy ontologies facing these problems:
  - To represent fuzzy DLs using crisp DLs
  - To reduce reasoning within fuzzy DLs to reasoning within crisp DLs
- This way it would be possible:
  - To translate them automatically into a crisp ontology language
  - To use currently available and highly optimized reasoners
- Not a lot of work following this line
  - U. Straccia showed a reasoning preserving procedure for $f\mathcal{ALCH}$

Motivation (4)

On the other hand, current fuzzy DLs still present some limitations which we think that should be overcome:

- Some works on fuzzy DLs deal with nominals (named individuals)
  - They choose not to fuzzify the nominal construct arguing that a fuzzy singleton set does not represent any real concept world.
  - Hence, only crisp concepts can be defined extensively, as nominals either have to fully belong to them or not
- There have been proposed fuzzy general concept inclusions which allow to constrain the truth value of a general concept inclusion
  - Current reasoning algorithms do not allow them

Contributions

1. We propose a different definition of $f\mathcal{SHOIN}$ including:
   - A fuzzy nominal construct
   - Reasoning with fuzzy GCIs
2. We reduce reasoning in $f_{KD}\mathcal{SHOIN}$ to reasoning in $\mathcal{SHOIN}$, extending the existing work on $f_{KD}\mathcal{ALCH}$

Agenda

Syntax: Knowledge base

- ABox: finite set of assertions about individuals:
  - Concept assertions $a : C$ (a is an instance of C)
  - Role assertions $(a, b) : R$ ((a, b) is an instance of R)
  - Individual assertions:
    - $a \neq b$ (a and b are different individuals)
    - $a = b$ (a and b refer to the same individual)
- TBox: a finite set of concept axioms:
  - General concept inclusions (GCI) $C \sqsubseteq D$ (C is more specific than D)
  - Concept definitions $C \equiv D$ ($C \sqsubseteq D$ and $D \sqsubseteq C$)
- RBox: a finite set of role axioms:
  - Role inclusions $R \sqsubseteq R'$ (R is more specific than R')
  - Role definitions $R \equiv R'$ ($R \sqsubseteq R'$ and $R' \sqsubseteq R$)
  - Transitive role axioms $trans(R)$ (R is transitive)

Syntax: Complex concepts and roles

- The concepts of the language can be built inductively:

$$
\begin{array}{rcl}
C, D \rightarrow & A & \mid \text{(atomic concept)} \\
& \top & \mid \text{(top concept)} \\
& \bot & \mid \text{(bottom concept)} \\
& C \sqcap D & \mid \text{(concept conjunction)} \\
& C \sqcup D & \mid \text{(concept disjunction)} \\
& \neg C & \mid \text{(concept negation)} \\
& \forall R.C & \mid \text{(universal quantification)} \\
& \exists R.C & \mid \text{(full existential quantification)} \\
& \{o_1, \dots, o_m\} & \mid \text{(nominals)} \\
& (\geq n\ S) & \mid \text{(at-least unqualified number restriction)} \\
& (\leq n\ S) & \text{(at-most unqualified number restriction)}
\end{array}
$$

- The roles of the language can be built using this syntax rule:

$$
\begin{array}{rcl}
R \rightarrow & R_A & \mid \text{(atomic role)} \\
& R^{-} & \text{(inverse role)}
\end{array}
$$

Semantics (1)

- An interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of:
  - A non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain)
  - An interpretation function $\cdot^{\mathcal{I}}$ mapping:
    - Every individual onto an element of $\Delta^{\mathcal{I}}$
    - Every atomic concept A onto a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
    - Every atomic role R onto a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.
- We do not impose unique name assumption (UNA), i.e. two nominals might refer to the same individual

Semantics (2)

- The interpretation is extended to complex concepts and roles by the following inductive definitions:

$$
\begin{array}{rcl}
\top^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} & = & \emptyset \\
(C \sqcap D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} & = & \{a : \forall b, (a,b) \notin R^{\mathcal{I}} \text{ or } b \in C^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} & = & \{a : \exists b, (a,b) \in R^{\mathcal{I}} \text{ and } b \in C^{\mathcal{I}}\} \\
\{o_1, \ldots, o_m\}^{\mathcal{I}} & = & \{o_1^{\mathcal{I}}, \ldots, o_m^{\mathcal{I}}\} \\
(\geq 0\ S.C)^{\mathcal{I}} & = & \top^{\mathcal{I}} = \Delta^{\mathcal{I}} \\
(\geq m^2\ S.C)^{\mathcal{I}} & = & \{a : |\{b : (a,b) \in S^{\mathcal{I}} \text{ and } b \in C^{\mathcal{I}}\}| \geq m\} \\
(\leq n\ S.C)^{\mathcal{I}} & = & \{a : |\{b : (a,b) \in S^{\mathcal{I}} \text{ and } b \in C^{\mathcal{I}}\}| \leq n\} \\
(R^-)^{\mathcal{I}} & = & \{(b,a) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | (a,b) \in R^{\mathcal{I}}\}
\end{array}
$$

[2] $m > 0$

Semantics (3)

- An interpretation $\mathcal{I}$ satisfies (is a model of):

$$
\begin{array}{rcl}
a : C & \text{iff} & a^{\mathcal{I}} \in C^{\mathcal{I}} \\
(a, b) : R & \text{iff} & (a, b)^{\mathcal{I}} \in R^{\mathcal{I}} \\
\langle a \neq b \rangle & \text{iff} & a^{\mathcal{I}} \neq b^{\mathcal{I}} \\
\langle a = b \rangle & \text{iff} & a^{\mathcal{I}} = b^{\mathcal{I}} \\
C \sqsubseteq D & \text{iff} & C^{\mathcal{I}} \subseteq D^{\mathcal{I}} \\
C \equiv D & \text{iff} & C^{\mathcal{I}} = D^{\mathcal{I}} \\
R \sqsubseteq R' & \text{iff} & R^{\mathcal{I}} \subseteq R'^{\mathcal{I}} \\
R \equiv R' & \text{iff} & R^{\mathcal{I}} = R'^{\mathcal{I}} \\
trans(R) & \text{iff} & (R)^{\mathcal{I}} \text{ is transitive} \\
\text{ABox } K_A & \text{iff} & \mathcal{I} \text{ satisfies each element in } K_A \\
\text{TBox } K_T & \text{iff} & \mathcal{I} \text{ satisfies each element in } K_T \\
\text{RBox } K_R & \text{iff} & \mathcal{I} \text{ satisfies each element in } K_R \\
\text{KB } K = \langle K_A, K_T, K_R \rangle & \text{iff} & \mathcal{I} \text{ satisfies all } K_A,\ K_T \text{ and } K_R
\end{array}
$$

Reasoning

- A DL not only stores axioms and assertions, but also offers some reasoning services, such as:
  - KB satisfiability
  - Concept satisfiability
  - Subsumption between concepts
  - Instance checking
- If a DL is closed under negation then all the basic reasoning services are reducible to KB satisfiability

Agenda

Syntax

- $f\mathcal{SHOIN}$ extends $\mathcal{SHOIN}$ to the fuzzy case by letting:
  - Concepts denote fuzzy sets of individuals
  - Roles denote fuzzy binary relations between individuals
- Our logic is similar to other approaches adding:
  - Fuzzy nominals
  - Constraints on fuzzy GCIs
- A fuzzy Knowledge Base (fKB) contains:
  - A fuzzy ABox fK_A
  - A fuzzy TBox fK_T
  - A fuzzy RBox fK_R

Syntax: fuzzy ABox

- Novelty: the truth value of constrained concept and role assertions
- Concepts and role assertions:
  - $\langle \Psi \geq \alpha \rangle$
  - $\langle \Psi > \beta \rangle$
  - $\langle \Phi \leq \beta \rangle$
  - $\langle \Phi < \alpha \rangle$
- where:
  - Ψ is an assertion of the form $a : C$ or $(a, b) : R$
  - Φ is an assertion of the form $a : C$
  - $\alpha \in (0, 1]$ (excludes 0)
  - $\beta \in [0, 1)$ (excludes 1)
- Some assertions are not allowed:
  - $\langle (a, b) : R \leq \beta \rangle, \langle (a, b) : R < \alpha \rangle$: relate to negated roles ($\notin \mathcal{SHOIN}$)
  - $\langle a : C > 1 \rangle, \langle a : C < 0 \rangle, \langle (a, b) : R > 1 \rangle$: trivially unsatisfiable
  - $\langle a : C \geq 0 \rangle, \langle a : C \leq 1 \rangle, \langle (a, b) : R \geq 0 \rangle$: trivially satisfiable
- Individual assertions:
  - $\langle a \neq b \rangle$
  - $\langle a = b \rangle$

Syntax: fuzzy TBox and RBox

- Novelty: the truth value of GCIs may be constrained
- Fuzzy GCIs:
  - $\langle C \sqsubseteq D \geq \alpha \rangle$
  - $\langle C \sqsubseteq D > \beta \rangle$
  - $\langle C \sqsubseteq D \leq \beta \rangle$
  - $\langle C \sqsubseteq D < \alpha \rangle$
- $C \equiv D$ is an abbreviation of $\langle C \sqsubseteq D \geq 1 \rangle$ and $\langle D \sqsubseteq C \geq 1 \rangle$
- Fuzzy role inclusions $R \sqsubseteq R'$
- Fuzzy role definitions $R \equiv R'$ ($R \sqsubseteq R'$ and $R' \sqsubseteq R$)
- Transitive role axiom $trans(R)$

Syntax: complex concepts and roles

- Concepts can be built inductively:

$$
\begin{array}{rcll}
C, D \rightarrow & A & | & \text{(atomic concept)} \\
& \top & | & \text{(top concept)} \\
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& C \sqcap D & | & \text{(concept conjunction)} \\
& C \sqcup D & | & \text{(concept disjunction)} \\
& \neg C & | & \text{(concept negation)} \\
& \forall R.C & | & \text{(universal quantification)} \\
& \exists R.C & | & \text{(full existential quantification)} \\
& \{(o_1, \alpha_1), \ldots, (o_m, \alpha_m)\} & | & \text{(nominals)} \\
& (\geq n\ S) & | & \text{(at-least number restriction)} \\
& (\leq n\ S) & & \text{(at-most number restriction)}
\end{array}
$$

- Complex roles can be built using this syntax rule: $R \rightarrow R_A \mid R^-$

Semantics (1)

- A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of:
  - A non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain)
  - A fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping:
    - Every individual onto an element of $\Delta^{\mathcal{I}}$
    - Every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0,1]$
    - Every role R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1]$
  - $C^{\mathcal{I}}$: membership degree function of the fuzzy concept C w.r.t. $\mathcal{I}$
  - $R^{\mathcal{I}}$: membership degree function of the fuzzy role R w.r.t. $\mathcal{I}$
- In fuzzy DLs most reasoning services are reducible to fKB satisfiability, so here in after we will only consider this task
- We do not impose unique name assumption (UNA)

Semantics (2)

- The fuzzy interpretation function is extended:

$$
\begin{aligned}
\top^{\mathcal{I}}(a) &= 1 \\
\bot^{\mathcal{I}}(a) &= 0 \\
(C \sqcap D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \wedge D^{\mathcal{I}}(a) \\
(C \sqcup D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \vee D^{\mathcal{I}}(a) \\
(\neg C)^{\mathcal{I}}(a) &= \neg C^{\mathcal{I}}(a) \\
(\forall R.C)^{\mathcal{I}}(a) &= \inf_{b \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a,b) \rightarrow C^{\mathcal{I}}(b)\} \\
(\exists R.C)^{\mathcal{I}}(a) &= \sup_{b \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a,b) \wedge C^{\mathcal{I}}(b)\} \\
\{(o_i, \alpha_i)\}^{\mathcal{I}}(a) &= \sup_{i \mid a \in \{o_i^{\mathcal{I}}\}} \alpha_i \\
(\geq 0)^{\mathcal{I}}(a) &= \top^{\mathcal{I}}(a) = 1 \\
(\geq m)^{\mathcal{I}}(a) &= \sup_{b_1, \ldots, b_m \in \Delta^{\mathcal{I}}} [\wedge_{i=1}^{m} S^{\mathcal{I}}(a, b_i) \bigwedge \wedge_{i<j} \{b_i \neq b_j\}] \\
(\leq n\ S)^{\mathcal{I}}(a) &= \neg(\geq n{+}1\ S)^{\mathcal{I}}(a) \\
(R^{-})^{\mathcal{I}}(a,b) &= R^{\mathcal{I}}(b,a)
\end{aligned}
$$

Fuzzy nominals

- Example: Country where German is a widely spoken language: $C \equiv \{germany, austria, switzerland\}$
- The classical semantics forces *switzerland* to fully belong to the concept or not: $\{o_i\}^{\mathcal{I}}(a) = 1$ if $a \in \{o_i^{\mathcal{I}}\}$ or 0 otherwise
- With fuzzy nominals: $\{(germany, 1), (austria, 1), (switzerland, 0.67)\}$
  - It does represent a real-life concept: a fuzzy set defined extensively
- Recall that the semantics is $\sup_{i \mid a \in \{o_i^{\mathcal{I}}\}} \alpha_i$
  - $a : C \leq 0.8$ prevents a of being *germany* or *austria*
  - Different from a fuzzy disjunction of nominals.
  - We consider equality between individuals ($a = o_i$) to be crisp
  - The definition generalizes the previous definition for nominals

- A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):

$\langle a : C \geq \alpha \rangle$[3]	iff	$C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \alpha$
$\langle (a, b) : R \geq \alpha \rangle$[3]	iff	$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \alpha$
$\langle a \neq b \rangle$	iff	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$
$\langle a = b \rangle$	iff	$a^{\mathcal{I}} = b^{\mathcal{I}}$
$\langle C \sqsubseteq D \geq \alpha \rangle$[3]	iff	$\inf_{a \in \Delta^{\mathcal{I}}} \{ C^{\mathcal{I}}(a) \rightarrow D^{\mathcal{I}}(a) \} \geq \alpha$
$C \equiv D$	iff	$C^{\mathcal{I}} = D^{\mathcal{I}}$
$R \sqsubseteq R'$	iff	$R^{\mathcal{I}} \subseteq R'^{\mathcal{I}}$
$R \equiv R'$	iff	$R^{\mathcal{I}} = R'^{\mathcal{I}}$
$trans(R)$	iff	$\forall a, b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a, b) \geq \sup_{c \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a, c) \wedge R^{\mathcal{I}}(c, b)$
ABox K_A	iff	$\mathcal{I}$ satisfies each element in K_A
TBox K_T	iff	$\mathcal{I}$ satisfies each element in K_T
RBox K_R	iff	$\mathcal{I}$ satisfies each element in K_R
fKB $\langle K_A, K_T, K_R \rangle$	iff	$\mathcal{I}$ satisfies all K_A, K_T and K_R

[3] Definitions are similar for $> \beta$, $\leq \beta$ and $< \alpha$

Fuzzy GCIs

- The definition of fuzzy GCIs allows concept subsumption to hold to a certain degree in $[0, 1]$
  - Example: $\langle \mathit{Inn} \sqsubseteq \mathit{Hotel} \geq 0.5 \rangle$
  - Translating universal quantification and GCIs to First Order Logic leads to implication functions
  - It seems natural to let both (or neither) of them be fuzzy
- This does not hold for role inclusion axioms
  - Asymmetry in the expressivity
  - The implication function would require the subjacent DL to have negated roles and role disjunction
  - We have preferred to consider $\mathcal{SHOIN}$, underlying OWL DL

Some properties (1)

Lemma

Fuzzy interpretations coincide with crisp interpretations if we restrict to the membership degrees of 0 and 1

- Here in after we concentrate on $f_{KD}\mathcal{SHOIN}$:

Gödel t-norm (minimum)	$\alpha \wedge \beta = \min\{\alpha, \beta\}$
Gödel t-conorm (maximum)	$\alpha \vee \beta = \max\{\alpha, \beta\}$
Łukasiewicz negation	$\neg\alpha = 1 - \alpha$
Kleene-Dienes implication	$\alpha \rightarrow \beta = \max\{1 - \alpha, \beta\}$

- This choice of the t-norm and the t-conorm eases the translation

- $f_{KD}\mathcal{SHOIN}$ allows some sort of modus ponens over concepts and roles, even with the new semantics of fuzzy GCIs:

Lemma

For $\alpha, \beta, \gamma \in [0,1]$, $\bowtie = \{\geq, >\}$ *and* $\alpha \neg\bowtie 1 - \beta$ $(\neg\geq = <, \neg> = \leq)$, *the following properties are verified:*

(i) $\langle a : C \bowtie \alpha \rangle$ *and* $\langle C \sqsubseteq D \bowtie \beta \rangle$ *imply* $\langle a : D \bowtie \beta \rangle$

(ii) $\langle (a,b) : R \bowtie \gamma \rangle$ *and* $\langle R \sqsubseteq R' \rangle$ *imply* $\langle (a,b) : R' \bowtie \gamma \rangle$

(iii) $\langle (a,b) : R \bowtie \alpha \rangle$ *and* $\langle a : \forall R.C \bowtie \beta \rangle$ *imply* $\langle b : C \bowtie \beta \rangle$

Some properties (3)

- The use of Kleene-Dienes implication in the semantics of fuzzy GCIs brings about two counter-intuitive effects:
  1. A concept does not fully subsume itself: $C \sqsubseteq C \Rightarrow \inf_{a \in \Delta^{\mathcal{I}}} \max\{1 - C^{\mathcal{I}}(a), C^{\mathcal{I}}(a)\} = 0.5$
  2. Crisp concept subsumption forces fuzzy concepts to be crisp: $\langle C \sqsubseteq D \geq 1 \rangle \Rightarrow \inf_{a \in \Delta^{\mathcal{I}}} \max\{1 - C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)\} \geq 1$ which is true iff for each element of the domain $D^{\mathcal{I}}(a) = 1$ or $C^{\mathcal{I}}(a) = 0$
- Need of further investigation involving alternative fuzzy operators!
  - A residuum based implication would fix 1: $a \rightarrow b = 1$ if $a \leq b$
  - Łukasiewicz implication would fix 2: $a \rightarrow b = \min\{1, 1 - a + b\}$

Agenda

Intuitive idea

- U. Straccia presents a reasoning preserving transformation for $f_{KD}\mathcal{ALCH}$ into crisp $\mathcal{ALCH}$
- We extend this work to $f_{KD}\mathcal{SHOIN}$
- Example: transform $\langle a : A \geq 0.8 \rangle$ into $a : A_{\geq 0.8}$ (0.8-cut of A)
- Procedure:
  1. Define some new atomic concepts and roles
  2. Add some new axioms to preserve the semantics of the fKB
  3. Map separately ABox, TBox and RBox

New elements (1)

- It has been shown that in $f_{KD}\mathcal{ALC}$, the set of the degrees which must be considered for any reasoning task can be computed as:

$$N^{fK} = X^{fK} \cup \{1 - \alpha \mid \alpha \in X^{fK}\}$$

- where X^{fK} is the set of degrees appearing in the fKB:

$$\begin{aligned} X^{fK} = \quad & \{0, 0.5, 1\} \cup \{\alpha \mid \langle \Psi \geq \alpha \rangle \in fK_A\} \cup \{\beta \mid \langle \Psi > \beta \rangle \in fK_A\} \\ & \cup \{1 - \beta \mid \langle \Phi \leq \beta \rangle \in fK_A\} \cup \{1 - \alpha \mid \langle \Phi < \alpha \rangle \in fK_A\} \\ & \cup \{\alpha \mid \langle \Omega \geq \alpha \rangle \in fK_T\} \cup \{\beta \mid \langle \Omega > \beta \rangle \in fK_T\} \\ & \cup \{1 - \beta \mid \langle \Omega \leq \beta \rangle \in fK_T\} \cup \{1 - \alpha \mid \langle \Omega < \alpha \rangle \in fK_T\} \end{aligned}$$

- This also holds in $f_{KD}\mathcal{SHOIN}$
- When other fuzzy operators are considered this is no longer true
  - We may calculate all possible degrees in $[0, 1]$ with a given precision, but further investigation is required

New elements (2)

- For every atomic concept and role in the fKB and for each degree $\in N^{fK}$ we create:
  - Four new atomic concepts $A_{\geq\alpha}, A_{>\beta}, A_{\leq\beta}, A_{<\alpha}$
  - Two new atomic roles $R_{\geq\alpha}, R_{>\beta}$
- Informally, $A_{\geq\alpha}$ represents the crisp set of individuals which are instance of A with degree higher or equal than α (α-cut of A)
- $A_{<0}, A_{>1}, R_{>1}$ are not considered (they are always empty sets)
- $A_{\leq 1}, A_{\geq 0}, R_{\geq 0}$ are not considered (they are equivalent to $\top$)

New elements (3)

- The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms
- New concept axioms:

$$\begin{array}{rcl} A_{\geq \gamma_{i+1}} & \sqsubseteq & A_{>\gamma_i} \\ A_{<\gamma_j} & \sqsubseteq & A_{\leq \gamma_j} \\ A_{\geq \gamma_j} \sqcap A_{<\gamma_j} & \sqsubseteq & \perp \\ \top & \sqsubseteq & A_{\geq \gamma_j} \sqcup A_{<\gamma_j} \end{array} \qquad \begin{array}{rcl} A_{>\gamma_i} & \sqsubseteq & A_{\geq \gamma_i} \\ A_{\leq \gamma_i} & \sqsubseteq & A_{<\gamma_{i+1}} \\ A_{>\gamma_i} \sqcap A_{\leq \gamma_i} & \sqsubseteq & \perp \\ \top & \sqsubseteq & A_{>\gamma_i} \sqcup A_{\leq \gamma_i} \end{array}$$

- New role axioms:

$$\begin{array}{rcl} R_{\geq \gamma_{i+1}} & \sqsubseteq & R_{>\gamma_i} \\ R_{>\gamma_i} & \sqsubseteq & R_{\geq \gamma_i} \end{array}$$

- $\langle (a,b) : R \leq \beta \rangle, \langle (a,b) : R < \alpha \rangle$ would need additional role constructs: $\sqcap_R, \sqcup_R, \top_R, \perp_R$

Mapping the ABox

- A fuzzy ABox is mapped into using a mapping σ:

$$\begin{array}{rcl}\sigma(\langle a : C \bowtie \gamma \rangle) & = & a : \rho(C, \bowtie \gamma) \\ \sigma(\langle (a,b) : R \bowtie \gamma \rangle) & = & (a,b) : \rho(R, \bowtie \gamma) \\ \sigma(\langle a \neq b \rangle) & = & a \neq b \\ \sigma(\langle a = b \rangle) & = & a = b\end{array}$$

- Fuzzy assertions are mapped into crisp assertions
  - ρ is inductively defined on the structure of concepts and roles

Mapping concepts and roles (1)

x	y	$\rho(x, y)$
A	$\geq \gamma$	$A_{\geq\gamma}$ if $\gamma \neq 0, \top$ otherwise
A	$> \gamma$	$A_{>\gamma}$, if $\gamma \neq 1, \bot$ otherwise
A	$\leq \gamma$	$A_{\leq\gamma}$ if $\gamma \neq 0, \top$ otherwise
A	$< \gamma$	$A_{<\gamma}$, if $\gamma \neq 1, \bot$ otherwise
R	$\geq \gamma$	$R_{\geq\gamma}$ if $\gamma \neq 0, \top$ otherwise
R	$> \gamma$	$R_{>\gamma}$, if $\gamma \neq 1, \bot$ otherwise
$\top$	$\geq \gamma$	$\top$
$\top$	$> \gamma$	$\top$ if $\gamma \neq 1, \bot$ otherwise
$\top$	$\leq \gamma$	$\top$ if $\gamma = 1, \bot$ otherwise
$\top$	$< \gamma$	$\bot$
$\bot$	$\geq \gamma$	$\top$ if $\gamma = 0, \bot$ otherwise
$\bot$	$> \gamma$	$\bot$
$\bot$	$\leq \gamma$	$\top$
$\bot$	$< \gamma$	$\top$ if $\gamma \neq 0, \bot$ otherwise

Mapping concepts and roles (2)

x	y	$\rho(x, y)$
$C \sqcap D$	$\{\geq, >\}\ \gamma$	$\rho(C, \{\geq, >\}\ \gamma) \sqcap \rho(D, \{\geq, >\}\ \gamma)$
$C \sqcap D$	$\{\leq, <\}\ \gamma$	$\rho(C, \{\leq, <\}\ \gamma) \sqcup \rho(D, \{\leq, <\}\ \gamma)$
$C \sqcup D$	$\{\geq, >\}\ \gamma$	$\rho(C, \{\geq, >\}\ \gamma) \sqcup \rho(D, \{\geq, >\}\ \gamma)$
$C \sqcup D$	$\{\leq, <\}\ \gamma$	$\rho(C, \{\leq, <\}\ \gamma \sqcap \rho(D, \{\leq, <\}\ \gamma)$
$\neg C$	$\{\geq, >\}\ \gamma$	$\rho(C, \{\leq, <\}\ 1 - \gamma)$
$\neg C$	$\{\leq, <\}\ \gamma$	$\rho(C, \{\geq, >\}\ 1 - \gamma)$
$\exists R.C$	$\{\geq, >\}\ \gamma$	$\exists \rho(R, \{\geq, >\}\ \gamma).\rho(C, \{\geq, >\}\ \gamma)$
$\exists R.C$	$\{\leq, <\}\ \gamma$	$\forall \rho(R, \{>, \geq\}\ \gamma).\rho(C, \{\leq, <\}\ \gamma)$
$\forall R.C$	$\{\geq, >\}\ \gamma$	$\forall \rho(R, \{>, \geq\}\ 1 - \gamma).\rho(C, \{\geq, >\}\ \gamma)$
$\forall R.C$	$\{\leq, <\}\ \gamma$	$\exists \rho(R, \{\geq, >\}\ 1 - \gamma).\rho(C, \{\leq, <\}\ \gamma)$

Mapping concepts and roles (3)

x	y	$\rho(x, y)$
$\{(o_1, \alpha_1), \ldots, (o_m, \alpha_m)\}$	$\bowtie \gamma$	$\{o_i \mid \alpha_i \bowtie \gamma, 1 \leq i \leq n\}_{\bowtie\gamma}$
$\geq 0\ S$	$\bowtie \gamma$	$\rho(\top, \bowtie \gamma)$
$\geq m\ S$	$\{\geq, >\}\ \gamma$	$\geq m\ \rho(S, \{\geq, >\}\ \gamma)$
$\geq m\ S$	$\{\leq, <\}\ \gamma$	$\leq m{-}1\ \rho(S, \{>, \geq\}\ \gamma)$
$\leq n\ S$	$\{\geq, >\}\ \gamma$	$\leq n\ \rho(S, \{>, \geq\}\ 1 - \gamma)$
$\leq n\ S$	$\{\leq, <\}\ \gamma$	$\geq n{+}1\ \rho(S, \{\geq, >\}\ 1 - \gamma)$
R^-	$\bowtie \gamma$	$\rho(R, \bowtie \gamma)^-$

Mapping the TBox

- A positive GCI ($\geq$, $>$) is reduced into a GCI:

$$\begin{aligned} k(\langle C \sqsubseteq D \geq \gamma \rangle) = & \quad \rho(C, > 1-\gamma) \sqsubseteq \rho(D, \geq \gamma) \\ k(\langle C \sqsubseteq D > \gamma \rangle) = & \quad \rho(C, \geq 1-\gamma) \sqsubseteq \rho(D, > \gamma) \end{aligned}$$

- A negative GCI ($\leq$, $<$) is reduced into an assertion about a new individual x:

$$\begin{aligned} A(\langle C \sqsubseteq D \leq \gamma \rangle) = & \quad x : \rho(C, \geq 1-\gamma) \sqcap \rho(D, \leq \gamma) \\ A(\langle C \sqsubseteq D < \gamma \rangle) = & \quad x : \rho(C, > 1-\gamma) \sqcap \rho(D, < \gamma) \end{aligned}$$

  - The natural reduction would be to a negated GCI, but it is not part of crisp $\mathcal{SHOIN}$
- How to deal with alternative implication functions?

Mapping the RBox

- A fuzzy RBox is reduced using a function $k(fK, fK_R) = \bigcup_{\Omega \in fK_R} k(\Omega)$
- Role axioms are reduced using a function $k(\Omega)$:

$$
\begin{aligned}
k(R \sqsubseteq R') &= \bigcup_{\gamma \in N^{fK}, \bowtie \in \{\geq, >\}} \rho(R, \bowtie \gamma) \sqsubseteq \rho(R', \bowtie \gamma) \\
k(trans(R)) &= \bigcup_{\gamma \in N^{fK}, \bowtie \in \{\geq, >\}} trans(\rho(R, \bowtie \gamma))
\end{aligned}
$$

Complexity

- A fKB fK is reduced into a KB $K(fK)$:

fKB	$K(fK)$
fK_A	$\sigma(fK_A) \cup A(fK_T)$
fK_T	$T(N^{fK}) \cup k(fK, fK_T)$
fK_R	$R(N^{fK}) \cup k(fK, fK_R)$

- The complexity is quadratic:
  - ABox is linear
  - TBox and RBox are quadratic
- In the previous work, a fuzzy GCI is reduced into a set of crisp GCIs
  - Our semantics for fuzzy GCIs allows to reduce each axiom into either an axiom or an assertion
  - This reduction in the size of the TBox is very interesting since reasoning with GCIs is a source of computational complexity

Theorem

A $f_{KD}\mathcal{SHOIN}$ fKB fK is satisfiable iff $K(fK)$ is satisfiable

- Firstly, it has to be proved that the translation preserves the satisfiability of every single statement of the fKB
  - If there exists a fuzzy interpretation satisfying a fuzzy statement, then a crisp interpretation satisfying the result of its translation can be built
- Secondly, it has to be proved that the translation preserves the satisfiability of the whole fKB
  - Clashes produced by pairs of conjugated axioms are preserved by the new concept axioms: $A_{\geq\gamma_j} \sqcap A_{<\gamma_j} \sqsubseteq \bot$, $A_{>\gamma_i} \sqcap A_{\leq\gamma_i} \sqsubseteq \bot$

Agenda

Conclusions

- A sound fuzzy extension of $\mathcal{SHOIN}$ including:
  - Fuzzy nominals, enabling to define fuzzy sets extensively
  - Reasoning with fuzzy GCIs, allowing to constrain the truth value of a GCI
- Reasoning preserving procedure into a crisp KB.
- Alternative approach to achieve fuzzy ontologies, reusing currently existing crisp ontology languages and reasoners
- The semantics of fuzzy GCIs:
  - Allows fuzzy GCIs holding to some degree
  - Reduces the size of the resulting TBox w.r.t. Zadeh implication
  - However, it imposes some counter-intuitive effects

Future work

- Empirical evaluation to test the translation
- Consider different fuzzy operators
  - In particular, avoid the counter-intuitive effects of the Kleene-Dienes implication
- Include a crisp representation for fuzzy datatypes
  - Since OWL does not currently allow to define customised datatypes, consider OWL Eu
- Consider more expressive DLs and, in particular, $\mathcal{SROIQ}$:
  - Subjacent DL of OWL 1.1
  - The additional expressivity in roles may help to overcome the asymmetry in fuzzy concept and role inclusion axioms

Questions?

Thank you very much for your attention