A Crisp Representation for Fuzzy $\mathcal{SHOIN}$ with Fuzzy Nominals and General Concept Inclusions

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Agenda

1. Introduction

2. A Quick Survey on SHOIN

3. Fuzzy SHOIN

4. A Crisp Representation for Fuzzy SHOIN

5. Conclusions and Future Work
Preliminaries

- **Ontologies** are a core element in the layered architecture of the Semantic Web.
- **Description Logics** (DLs) are a family of logics for representing structured knowledge.
- DLs have been proved to be very useful as ontology languages.
- Standard for ontology representation: **OWL** Web Ontology Language:
  - OWL Lite
  - OWL DL, the most used, nearly equivalent to \( SHOIN(D) \) DL
  - OWL Full, undecidable
Motivation

- Classical ontologies/DLs are not appropriate for **uncertain or imprecise knowledge**

- Examples:
  - German is **generally** spoken in Germany, Austria and Switzerland
  - An inn is a **cheap** and **small** hotel

- Vagueness is inherent to a lot of real-world application domains
  - The Semantic Web will not be fully operative as long as it does not provide means to manage it

- One solution: **Fuzzy DLs** (DLs extended with fuzzy sets theory)
Motivation (2)

- OWL may be extended to a fuzzy DL-based language e.g. FuzzyOWL
  - The large number of resources available should be adapted
  - In particular, we need reasoners

- Reasoning within expressive DLs has a very high worst-case complexity
  - Significant gap between the design of a decision procedure and the achievement of a practical implementation
  - The experience with crisp DLs induces us to think that developing highly optimized implementations will be a hard task where ad-hoc mechanisms should be used for every particular fuzzy DL
  - In fact, there is no implemented reasoner for $fSHOIN$
  - Nothing is known about the efficiency of existing reasoners:
    - fuzzyDL ($f_{KD}^{1}SHIF(D), f_{L}SHIF(D)$)
    - Fire ($f_{KD}SHIN'$)

1 subscript stands for the implication used, which specifies the other operators.
Alternative way to obtain fuzzy ontologies facing these problems:

- To represent fuzzy DLs using crisp DLs
- To reduce reasoning within fuzzy DLs to reasoning within crisp DLs

This way it would be possible:

- To translate them automatically into a crisp ontology language
- To use currently available and highly optimized reasoners

Not a lot of work following this line

- U. Straccia showed a reasoning preserving procedure for \textit{fALCH}
On the other hand, current fuzzy DLs still present some limitations which we think that should be overcome:

- Some works on fuzzy DLs deal with nominals (named individuals)
  - They choose not to fuzzify the nominal construct arguing that a fuzzy singleton set does not represent any real concept world.
  - Hence, only crisp concepts can be defined extensively, as nominals either have to fully belong to them or not

- There have been proposed fuzzy general concept inclusions which allow to constrain the truth value of a general concept inclusion
  - Current reasoning algorithms do not allow them
Contributions

1. We propose a different definition of $f\text{SHOIN}$ including:
   - A fuzzy nominal construct
   - Reasoning with fuzzy GCIs

2. We reduce reasoning in $f_{KD}\text{SHOIN}$ to reasoning in $\text{SHOIN}$, extending the existing work on $f_{KD}\text{ALCH}$
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Syntax: Knowledge base

**ABox**: finite set of assertions about individuals:
- Concept assertions $a : C$ ($a$ is an instance of $C$)
- Role assertions $(a, b) : R$ ($(a, b)$ is an instance of $R$)
- Individual assertions:
  - $a \neq b$ ($a$ and $b$ are different individuals)
  - $a = b$ ($a$ and $b$ refer to the same individual)

**TBox**: a finite set of concept axioms:
- General concept inclusions (GCI) $C \sqsubseteq D$ ($C$ is more specific than $D$)
- Concept definitions $C \equiv D$ ($C \sqsubseteq D$ and $D \sqsubseteq C$)

**RBox**: a finite set of role axioms:
- Role inclusions $R \sqsubseteq R'$ ($R$ is more specific than $R'$)
- Role definitions $R \equiv R'$ ($R \sqsubseteq R'$ and $R' \sqsubseteq R$)
- Transitive role axioms $\text{trans}(R)$ ($R$ is transitive)
The concepts of the language can be built inductively:

- $C, D \rightarrow A$ (atomic concept)
- $\top$ (top concept)
- $\bot$ (bottom concept)
- $C \sqcap D$ (concept conjunction)
- $C \sqcup D$ (concept disjunction)
- $\neg C$ (concept negation)
- $\forall R.C$ (universal quantification)
- $\exists R.C$ (full existential quantification)
- $\{o_1, \ldots, o_m\}$ (nominals)
- $(\geq n S)$ (at-least unqualified number restriction)
- $(\leq n S)$ (at-most unqualified number restriction)

The roles of the language can be built using this syntax rule:

- $R \rightarrow R_A$ (atomic role)
- $R^-$ (inverse role)
An interpretation $\mathcal{I}$ is a pair $(\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consisting of:

- A non empty set $\Delta^\mathcal{I}$ (the interpretation domain)
- An interpretation function $\cdot^\mathcal{I}$ mapping:
  - Every individual onto an element of $\Delta^\mathcal{I}$
  - Every atomic concept $A$ onto a set $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$
  - Every atomic role $R$ onto a binary relation $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$.

We do not impose unique name assumption (UNA), i.e. two nominals might refer to the same individual.
The interpretation is extended to complex concepts and roles by the following inductive definitions:

\[ \top^\mathcal{I} = \Delta^\mathcal{I} \]
\[ \bot^\mathcal{I} = \emptyset \]
\[ (C \cap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I} \]
\[ (C \cup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I} \]
\[ (\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I} \]
\[ (\forall R.C)^\mathcal{I} = \{ a : \forall b, (a, b) \notin R^\mathcal{I} \text{ or } b \in C^\mathcal{I} \} \]
\[ (\exists R.C)^\mathcal{I} = \{ a : \exists b, (a, b) \in R^\mathcal{I} \text{ and } b \in C^\mathcal{I} \} \]
\[ \{ o_1, \ldots, o_m \}^\mathcal{I} = \{ o_1^\mathcal{I}, \ldots, o_m^\mathcal{I} \} \]
\[ (\geq 0 \ S.C)^\mathcal{I} = \top^\mathcal{I} = \Delta^\mathcal{I} \]
\[ (\geq m^2 \ S.C)^\mathcal{I} = \{ a : \{ b : (a, b) \in S^\mathcal{I} \text{ and } b \in C^\mathcal{I} \} \geq m \} \]
\[ (\leq n \ S.C)^\mathcal{I} = \{ a : \{ b : (a, b) \in S^\mathcal{I} \text{ and } b \in C^\mathcal{I} \} \leq n \} \]
\[ (R^-)^\mathcal{I} = \{(b, a) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} | (a, b) \in R^\mathcal{I} \} \]
An interpretation \( \mathcal{I} \) satisfies (is a model of):

- \( a : C \) iff \( a^\mathcal{I} \in C^\mathcal{I} \)
- \( (a, b) : R \) iff \( (a, b)^\mathcal{I} \in R^\mathcal{I} \)
- \( \langle a \neq b \rangle \) iff \( a^\mathcal{I} \neq b^\mathcal{I} \)
- \( \langle a = b \rangle \) iff \( a^\mathcal{I} = b^\mathcal{I} \)
- \( C \sqsubseteq D \) iff \( C^\mathcal{I} \subseteq D^\mathcal{I} \)
- \( C \equiv D \) iff \( C^\mathcal{I} = D^\mathcal{I} \)
- \( R \sqsubseteq R' \) iff \( R^\mathcal{I} \subseteq R'^\mathcal{I} \)
- \( R \equiv R' \) iff \( R^\mathcal{I} = R'^\mathcal{I} \)
- \( \text{trans}(R) \) iff \( (R)^\mathcal{I} \) is transitive
- \( \text{ABox } K_A \) iff \( \mathcal{I} \) satisfies each element in \( K_A \)
- \( \text{TBox } K_T \) iff \( \mathcal{I} \) satisfies each element in \( K_T \)
- \( \text{RBox } K_R \) iff \( \mathcal{I} \) satisfies each element in \( K_R \)
- \( \text{KB } K = \langle K_A, K_T, K_R \rangle \) iff \( \mathcal{I} \) satisfies all \( K_A, K_T \) and \( K_R \)
A DL not only stores axioms and assertions, but also offers some reasoning services, such as:

- KB satisfiability
- Concept satisfiability
- Subsumption between concepts
- Instance checking

If a DL is closed under negation then all the basic reasoning services are reducible to KB satisfiability.
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**fSHOIN** extends **SHOIN** to the fuzzy case by letting:
- Concepts denote fuzzy sets of individuals
- Roles denote fuzzy binary relations between individuals

Our logic is similar to other approaches adding:
- Fuzzy nominals
- Constraints on fuzzy GCIs

A fuzzy Knowledge Base (fKB) contains:
- A fuzzy ABox \( fK_A \)
- A fuzzy TBox \( fK_T \)
- A fuzzy RBox \( fK_R \)
Novelty: the truth value of constrained concept and role assertions

Concepts and role assertions:

- $\langle \psi \geq \alpha \rangle$
- $\langle \psi > \beta \rangle$
- $\langle \phi \leq \beta \rangle$
- $\langle \phi < \alpha \rangle$

where:

- $\psi$ is an assertion of the form $a : C$ or $(a, b) : R$
- $\phi$ is an assertion of the form $a : C$
- $\alpha \in (0, 1]$ (excludes 0)
- $\beta \in [0, 1)$ (excludes 1)

Some assertions are not allowed:

- $\langle (a, b) : R \leq \beta \rangle$, $\langle (a, b) : R < \alpha \rangle$: relate to negated roles ($\not\in \text{SHOIN}$)
- $\langle a : C > 1 \rangle$, $\langle a : C < 0 \rangle$, $\langle (a, b) : R > 1 \rangle$: trivially unsatisfiable
- $\langle a : C \geq 0 \rangle$, $\langle a : C \leq 1 \rangle$, $\langle (a, b) : R \geq 0 \rangle$: trivially satisfiable

Individual assertions:

- $\langle a \neq b \rangle$
- $\langle a = b \rangle$
Novelty: the truth value of GCIs may be constrained

Fuzzy GCIs:
- $\langle C \sqsubseteq D \geq \alpha \rangle$
- $\langle C \sqsubseteq D > \beta \rangle$
- $\langle C \sqsubseteq D \leq \beta \rangle$
- $\langle C \sqsubseteq D < \alpha \rangle$

$C \equiv D$ is an abbreviation of $\langle C \sqsubseteq D \geq 1 \rangle$ and $\langle D \sqsubseteq C \geq 1 \rangle$

Fuzzy role inclusions $R \sqsubseteq R'$

Fuzzy role definitions $R \equiv R'$ ($R \sqsubseteq R'$ and $R' \sqsubseteq R$)

Transitive role axiom $\text{trans}(R)$
Syntax: complex concepts and roles

- Concepts can be built inductively:

\[ C, D \rightarrow \]

\[ A \quad | \quad \text{atomic concept} \]
\[ \top \quad | \quad \text{top concept} \]
\[ \bot \quad | \quad \text{bottom concept} \]
\[ C \sqcap D \quad | \quad \text{concept conjunction} \]
\[ C \sqcup D \quad | \quad \text{concept disjunction} \]
\[ \neg C \quad | \quad \text{concept negation} \]
\[ \forall R. C \quad | \quad \text{universal quantification} \]
\[ \exists R. C \quad | \quad \text{full existential quantification} \]
\[ \{(o_1, \alpha_1), \ldots, (o_m, \alpha_m)\} \quad | \quad \text{nominals} \]
\[ (\geq n S) \quad | \quad \text{at-least number restriction} \]
\[ (\leq n S) \quad | \quad \text{at-most number restriction} \]

- Complex roles can be built using this syntax rule: \( R \rightarrow R_A \mid R^- \)
Semantics (1)

- A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consisting of:
  - A non empty set $\Delta^\mathcal{I}$ (the interpretation domain)
  - A fuzzy interpretation function $\cdot^\mathcal{I}$ mapping:
    - Every individual onto an element of $\Delta^\mathcal{I}$
    - Every concept $C$ onto a function $C^\mathcal{I} : \Delta^\mathcal{I} \rightarrow [0, 1]$
    - Every role $R$ onto a function $R^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow [0, 1]$
  - $C^\mathcal{I}$: membership degree function of the fuzzy concept $C$ w.r.t. $\mathcal{I}$
  - $R^\mathcal{I}$: membership degree function of the fuzzy role $R$ w.r.t. $\mathcal{I}$

- In fuzzy DLs most reasoning services are reducible to fKB satisfiability, so here in after we will only consider this task

- We do not impose unique name assumption (UNA)
The fuzzy interpretation function is extended:

\[
\begin{align*}
\top^\mathcal{I}(a) &= 1 \\
\bot^\mathcal{I}(a) &= 0 \\
(C \sqcap D)^\mathcal{I}(a) &= C^\mathcal{I}(a) \land D^\mathcal{I}(a) \\
(C \sqcup D)^\mathcal{I}(a) &= C^\mathcal{I}(a) \lor D^\mathcal{I}(a) \\
(\neg C)^\mathcal{I}(a) &= \neg C^\mathcal{I}(a) \\
(\forall R. C)^\mathcal{I}(a) &= \inf_{b \in \Delta^\mathcal{I}} \{R^\mathcal{I}(a, b) \rightarrow C^\mathcal{I}(b)\} \\
(\exists R. C)^\mathcal{I}(a) &= \sup_{b \in \Delta^\mathcal{I}} \{R^\mathcal{I}(a, b) \land C^\mathcal{I}(b)\} \\
\{(o_i, \alpha_i)\}^\mathcal{I}(a) &= \sup_{i \mid a \in \{o_i^\mathcal{I}\}} \alpha_i \\
(\geq 0)^\mathcal{I}(a) &= \top^\mathcal{I}(a) = 1 \\
(\geq m)^\mathcal{I}(a) &= \sup_{b_1, \ldots, b_m \in \Delta^\mathcal{I}} [\land_{i=1}^m S^\mathcal{I}(a, b_i) \land \land_{i<j} \{b_i \neq b_j\}] \\
(\leq n \, S)^\mathcal{I}(a) &= \neg(\geq n+1 \, S)^\mathcal{I}(a) \\
(R^-)^\mathcal{I}(a, b) &= R^\mathcal{I}(b, a)
\end{align*}
\]
Fuzzy nominals

- Example: Country where German is a widely spoken language: \( C \equiv \{ \text{germany, austria, switzerland} \} \)

- The classical semantics forces switzerland to fully belong to the concept or not: \( \{ o_i \}^T(a) = 1 \) if \( a \in \{ o_i^T \} \) or 0 otherwise

- With fuzzy nominals: \( \{(\text{germany}, 1), (\text{austria}, 1), (\text{switzerland}, 0.67)\} \)
  - It does represent a real-life concept: a fuzzy set defined extensively

- Recall that the semantics is \( \sup_i | a \in \{ o_i^T \} \alpha_i \)
  - \( a : C \leq 0.8 \) prevents \( a \) of being \( \text{germany} \) or \( \text{austria} \)
  - Different from a fuzzy disjunction of nominals.
  - We consider equality between individuals \( (a = o_i) \) to be crisp
  - The definition generalizes the previous definition for nominals
A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):

$$\langle a : C \geq \alpha \rangle^3 \text{ iff } C^\mathcal{I}(a^\mathcal{I}) \geq \alpha$$
$$\langle (a, b) : R \geq \alpha \rangle^3 \text{ iff } R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \geq \alpha$$
$$\langle a \neq b \rangle \text{ iff } a^\mathcal{I} \neq b^\mathcal{I}$$
$$\langle a = b \rangle \text{ iff } a^\mathcal{I} = b^\mathcal{I}$$

$$\langle C \sqsubseteq D \geq \alpha \rangle^3 \text{ iff } \inf_{a \in \Delta^\mathcal{I}} \{C^\mathcal{I}(a) \rightarrow D^\mathcal{I}(a)\} \geq \alpha$$
$$C \equiv D \text{ iff } C^\mathcal{I} = D^\mathcal{I}$$
$$R \sqsubseteq R' \text{ iff } R^\mathcal{I} \subseteq R'^\mathcal{I}$$
$$R \equiv R' \text{ iff } R^\mathcal{I} = R'^\mathcal{I}$$

trans($R$) iff $\forall a, b \in \Delta^\mathcal{I}, R^\mathcal{I}(a, b) \geq \sup_{c \in \Delta^\mathcal{I}} R^\mathcal{I}(a, c) \land R^\mathcal{I}(c, b)$

ABox $K_A$ iff $\mathcal{I}$ satisfies each element in $K_A$

TBox $K_T$ iff $\mathcal{I}$ satisfies each element in $K_T$

RBox $K_R$ iff $\mathcal{I}$ satisfies each element in $K_R$

fKB $\langle K_A, K_T, K_R \rangle$ iff $\mathcal{I}$ satisfies all $K_A$, $K_T$ and $K_R$

---

Definitions are similar for $> \beta$, $\leq \beta$ and $< \alpha$
The definition of fuzzy GCIs allows concept subsumption to hold to a certain degree in $[0, 1]$

- Example: $\langle Inn \sqsubseteq Hotel \geq 0.5 \rangle$
- Translating universal quantification and GCIs to First Order Logic leads to implication functions
- It seems natural to let both (or neither) of them be fuzzy

This does not hold for role inclusion axioms

- Asymmetry in the expressivity
- The implication function would require the subjacent DL to have negated roles and role disjunction
- We have preferred to consider $SHOIN^\bot$, underlying OWL DL
Some properties (1)

Lemma

Fuzzy interpretations coincide with crisp interpretations if we restrict to the membership degrees of 0 and 1

Here in after we concentrate on $f_{KD\text{SHOIN}}$:

- Gödel t-norm (minimum): $\alpha \land \beta = \min\{\alpha, \beta\}$
- Gödel t-conorm (maximum): $\alpha \lor \beta = \max\{\alpha, \beta\}$
- Łukasiewicz negation: $\neg \alpha = 1 - \alpha$
- Kleene-Dienes implication: $\alpha \rightarrow \beta = \max\{1 - \alpha, \beta\}$

This choice of the t-norm and the t-conorm eases the translation.
Some properties (2)

- $f_{KD}SHOIN$ allows some sort of modus ponens over concepts and roles, even with the new semantics of fuzzy GCIs:

**Lemma**

For $\alpha, \beta, \gamma \in [0, 1]$, $\succsim = \{\geq, >\}$ and $\alpha \not\succsim 1 - \beta$ ($\neg \geq = <, \neg > = \leq$), the following properties are verified:

1. $\langle a : C \succsim \alpha \rangle$ and $\langle C \sqsubseteq D \succsim \beta \rangle$ imply $\langle a : D \succsim \beta \rangle$
2. $\langle (a, b) : R \succsim \gamma \rangle$ and $\langle R \sqsubseteq R' \rangle$ imply $\langle (a, b) : R' \succsim \gamma \rangle$
3. $\langle (a, b) : R \succsim \alpha \rangle$ and $\langle a : \forall R.C \succsim \beta \rangle$ imply $\langle b : C \succsim \beta \rangle$
The use of Kleene-Dienes implication in the semantics of fuzzy GCIs brings about two counter-intuitive effects:

1. A concept does not fully subsume itself:
   \[ C \sqsubseteq C \Rightarrow \inf_{a \in \Delta_I} \max\{1 - C^I(a), C^I(a)\} = 0.5 \]

2. Crisp concept subsumption forces fuzzy concepts to be crisp:
   \[ \langle C \sqsubseteq D \geq 1 \rangle \Rightarrow \inf_{a \in \Delta_I} \max\{1 - C^I(a), D^I(a)\} \geq 1 \text{ which is true iff for each element of the domain } D^I(a) = 1 \text{ or } C^I(a) = 0 \]

Need of further investigation involving alternative fuzzy operators!

- A residuum based implication would fix 1: \[ a \rightarrow b = 1 \text{ if } a \leq b \]
- Łukasiewicz implication would fix 2: \[ a \rightarrow b = \min\{1, 1 - a + b\} \]
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U. Straccia presents a reasoning preserving transformation for $f_{KD}ALCH$ into crisp $ALCH$

We extend this work to $f_{KD}SHOIN$

Example: transform $\langle a : A \geq 0.8 \rangle$ into $a : A_{\geq 0.8}$ (0.8-cut of $A$)

Procedure:

1. Define some new atomic concepts and roles
2. Add some new axioms to preserve the semantics of the fKB
3. Map separately ABox, TBox and RBox
New elements (1)

- It has been shown that in $f_{KD\text{ALC}}$, the set of the degrees which must be considered for any reasoning task can be computed as:

$$N^{fK} = X^{fK} \cup \{1 - \alpha \mid \alpha \in X^{fK}\}$$

- where $X^{fK}$ is the set of degrees appearing in the fKB:

$$X^{fK} = \{0, 0.5, 1\} \cup \{\alpha \mid \langle \Psi \geq \alpha \rangle \in fK_A\} \cup \{\beta \mid \langle \Psi > \beta \rangle \in fK_A\} \cup \{1 - \beta \mid \langle \Phi \leq \beta \rangle \in fK_A\} \cup \{1 - \alpha \mid \langle \Phi < \alpha \rangle \in fK_A\} \cup \{\alpha \mid \langle \Omega \geq \alpha \rangle \in fK_T\} \cup \{\beta \mid \langle \Omega > \beta \rangle \in fK_T\} \cup \{1 - \beta \mid \langle \Omega \leq \beta \rangle \in fK_T\} \cup \{1 - \alpha \mid \langle \Omega < \alpha \rangle \in fK_T\}$$

- This also holds in $f_{KD\text{SHOIN}}$

- When other fuzzy operators are considered this is no longer true
  - We may calculate all possible degrees in $[0, 1]$ with a given precision, but further investigation is required
For every atomic concept and role in the fKB and for each degree \( \alpha \in N^{fK} \) we create:

- Four new atomic concepts \( A_{\geq \alpha}, A_{> \beta}, A_{\leq \beta}, A_{< \alpha} \)
- Two new atomic roles \( R_{\geq \alpha}, R_{> \beta} \)

Informally, \( A_{\geq \alpha} \) represents the crisp set of individuals which are instance of \( A \) with degree higher or equal than \( \alpha \) (\( \alpha \)-cut of \( A \))

- \( A_{< 0}, A_{> 1}, R_{> 1} \) are not considered (they are always empty sets)
- \( A_{\leq 1}, A_{\geq 0}, R_{\geq 0} \) are not considered (they are equivalent to \( \top \))
The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms

New concept axioms:

\[
\begin{align*}
A_{\geq \gamma_{i+1}} & \sqsubseteq A_{\geq \gamma_i} & A_{\geq \gamma_i} & \sqsubseteq A_{\geq \gamma_i} \\
A_{\leq \gamma_j} & \sqsubseteq A_{\leq \gamma_j} & A_{\leq \gamma_i} & \sqsubseteq A_{\leq \gamma_i} \\
A_{\geq \gamma_j} \sqcap A_{\leq \gamma_j} & \sqsubseteq \bot & A_{\geq \gamma_j} \sqcap A_{\leq \gamma_i} & \sqsubseteq \bot \\
\top & \sqsubseteq A_{\geq \gamma_j} \sqcup A_{\leq \gamma_j} & \top & \sqsubseteq A_{\geq \gamma_i} \sqcup A_{\leq \gamma_i}
\end{align*}
\]

New role axioms:

\[
\begin{align*}
R_{\geq \gamma_{i+1}} & \sqsubseteq R_{\geq \gamma_i} & R_{\geq \gamma_i} & \sqsubseteq R_{\geq \gamma_i} \\
R_{\leq \gamma_i} & \sqsubseteq R_{\leq \gamma_i} & R_{\leq \gamma_i} & \sqsubseteq R_{\leq \gamma_i}
\end{align*}
\]

\[\langle (a, b) : R \leq \beta \rangle, \langle (a, b) : R < \alpha \rangle\] would need additional role constructs: \(\sqcap_R, \sqcup_R, \top_R, \bot_R\)
A fuzzy ABox is mapped into using a mapping $\sigma$:

$$
\begin{align*}
\sigma(\langle a : C \Join \gamma \rangle) &= a : \rho(C, \Join \gamma) \\
\sigma(\langle (a, b) : R \Join \gamma \rangle) &= (a, b) : \rho(R, \Join \gamma) \\
\sigma(\langle a \not\equiv b \rangle) &= a \not\equiv b \\
\sigma(\langle a = b \rangle) &= a = b
\end{align*}
$$

- Fuzzy assertions are mapped into **crisp assertions**
  - $\rho$ is inductively defined on the structure of concepts and roles
Mapping concepts and roles (1)

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</tbody>
</table>
Mapping concepts and roles (2)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\rho(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C \sqcap D$</td>
<td>${\geq, &gt;} \gamma$</td>
<td>$\rho(C, {\geq, &gt;} \gamma) \sqcap \rho(D, {\geq, &gt;} \gamma)$</td>
</tr>
<tr>
<td>$C \sqcap D$</td>
<td>${\leq, &lt;} \gamma$</td>
<td>$\rho(C, {\leq, &lt;} \gamma) \sqcup \rho(D, {\leq, &lt;} \gamma)$</td>
</tr>
<tr>
<td>$C \sqcup D$</td>
<td>${\geq, &gt;} \gamma$</td>
<td>$\rho(C, {\geq, &gt;} \gamma) \sqcup \rho(D, {\geq, &gt;} \gamma)$</td>
</tr>
<tr>
<td>$C \sqcup D$</td>
<td>${\leq, &lt;} \gamma$</td>
<td>$\rho(C, {\leq, &lt;} \gamma) \sqcap \rho(D, {\leq, &lt;} \gamma)$</td>
</tr>
<tr>
<td>$\neg C$</td>
<td>${\geq, &gt;} \gamma$</td>
<td>$\rho(C, {\leq, &lt;} 1 - \gamma)$</td>
</tr>
<tr>
<td>$\neg C$</td>
<td>${\leq, &lt;} \gamma$</td>
<td>$\rho(C, {\geq, &gt;} 1 - \gamma)$</td>
</tr>
<tr>
<td>$\exists R.C$</td>
<td>${\geq, &gt;} \gamma$</td>
<td>$\exists \rho(R, {\geq, &gt;} \gamma).\rho(C, {\geq, &gt;} \gamma)$</td>
</tr>
<tr>
<td>$\exists R.C$</td>
<td>${\leq, &lt;} \gamma$</td>
<td>$\forall \rho(R, {&gt;, \geq} \gamma).\rho(C, {\leq, &lt;} \gamma)$</td>
</tr>
<tr>
<td>$\forall R.C$</td>
<td>${\geq, &gt;} \gamma$</td>
<td>$\forall \rho(R, {&gt;, \geq} 1 - \gamma).\rho(C, {\geq, &gt;} \gamma)$</td>
</tr>
<tr>
<td>$\forall R.C$</td>
<td>${\leq, &lt;} \gamma$</td>
<td>$\exists \rho(R, {\geq, &gt;} 1 - \gamma).\rho(C, {\leq, &lt;} \gamma)$</td>
</tr>
</tbody>
</table>
### Mapping concepts and roles (3)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\rho(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${(o_1, \alpha_1), \ldots, (o_m, \alpha_m)}$</td>
<td>$\otimes \gamma$</td>
<td>${o_i \mid \alpha_i \otimes \gamma, 1 \leq i \leq n}_{\otimes \gamma}$</td>
</tr>
<tr>
<td>$\geq 0$ S</td>
<td>$\otimes \gamma$</td>
<td>$\rho(T, \otimes \gamma)$</td>
</tr>
<tr>
<td>$\geq m$ S</td>
<td>${\geq, &gt;} \gamma$</td>
<td>$\geq m \rho(S, {\geq, &gt;} \gamma)$</td>
</tr>
<tr>
<td>$\geq m$ S</td>
<td>${\leq, &lt;} \gamma$</td>
<td>$\leq m-1 \rho(S, {&gt;, \geq} \gamma)$</td>
</tr>
<tr>
<td>$\leq n$ S</td>
<td>${\geq, &gt;} \gamma$</td>
<td>$\leq n \rho(S, {&gt;, \geq} 1 - \gamma)$</td>
</tr>
<tr>
<td>$\leq n$ S</td>
<td>${\leq, &lt;} \gamma$</td>
<td>$\geq n+1 \rho(S, {\geq, &gt;} 1 - \gamma)$</td>
</tr>
<tr>
<td>$R^-$</td>
<td>$\otimes \gamma$</td>
<td>$\rho(R, \otimes \gamma)^-$</td>
</tr>
</tbody>
</table>
Mapping the TBox

- A positive GCI ($\geq, >$) is reduced into a GCI:

\[
k(\langle C \sqsubseteq D \geq \gamma \rangle) = \rho(C, > 1 - \gamma) \sqsubseteq \rho(D, \geq \gamma)
\]
\[
k(\langle C \sqsubseteq D > \gamma \rangle) = \rho(C, \geq 1 - \gamma) \sqsubseteq \rho(D, > \gamma)
\]

- A negative GCI ($\leq, <$) is reduced into an assertion about a new individual $x$:

\[
A(\langle C \sqsubseteq D \leq \gamma \rangle) = x : \rho(C, \geq 1 - \gamma) \sqcap \rho(D, \leq \gamma)
\]
\[
A(\langle C \sqsubseteq D < \gamma \rangle) = x : \rho(C, > 1 - \gamma) \sqcap \rho(D, < \gamma)
\]

- The natural reduction would be to a negated GCI, but it is not part of crisp $SHOIN$.

- How to deal with alternative implication functions?
Mapping the RBox

- A fuzzy RBox is reduced using a function $k(fK, fK_R) = \bigcup_{\Omega \in fK_R} k(\Omega)$
- Role axioms are reduced using a function $k(\Omega)$:

  \[
  k(R \sqsubseteq R') = \bigcup_{\gamma \in N^{fK}, \in\{\geq,>\}} \rho(R, \otimes \gamma) \sqsubseteq \rho(R', \otimes \gamma)
  \]

  \[
  k(\text{trans}(R)) = \bigcup_{\gamma \in N^{fK}, \in\{\geq,>\}} \text{trans}(\rho(R, \otimes \gamma))
  \]
Complexity

- A fKB $fK$ is reduced into a KB $K(fK)$:

<table>
<thead>
<tr>
<th>$fKB$</th>
<th>$K(fK)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$fK_A$</td>
<td>$\sigma(fK_A) \cup A(fK_T)$</td>
</tr>
<tr>
<td>$fK_T$</td>
<td>$T(N^{fK}) \cup k(fK, fK_T)$</td>
</tr>
<tr>
<td>$fK_R$</td>
<td>$R(N^{fK}) \cup k(fK, fK_R)$</td>
</tr>
</tbody>
</table>

- The complexity is quadratic:
  - ABox is linear
  - TBox and RBox are quadratic

- In the previous work, a fuzzy GCI is reduced into a set of crisp GCIs
  - Our semantics for fuzzy GCIs allows to reduce each axiom into either an axiom or an assertion
  - This reduction in the size of the TBox is very interesting since reasoning with GCIs is a source of computational complexity
Reasoning

**Theorem**

A $f_{KD}SHOIN$ fKB $fK$ is satisfiable iff $K(fK)$ is satisfiable

- Firstly, it has to be proved that the translation preserves the satisfiability of every single statement of the fKB
  - If there exists a fuzzy interpretation satisfying a fuzzy statement, then a crisp interpretation satisfying the result of its translation can be built

- Secondly, it has to be proved that the translation preserves the satisfiability of the whole fKB
  - Clashes produced by pairs of conjugated axioms are preserved by the new concept axioms: $A_{\geq \gamma_j} \sqcap A_{< \gamma_j} \sqsubseteq \bot$, $A_{> \gamma_i} \sqcap A_{\leq \gamma_i} \sqsubseteq \bot$
Agenda

1. Introduction
2. A Quick Survey on \textit{SHOIN}
3. Fuzzy \textit{SHOIN}
4. A Crisp Representation for Fuzzy \textit{SHOIN}
5. Conclusions and Future Work
Conclusions

- A sound **fuzzy extension of SHOIN** including:
  - Fuzzy nominals, enabling to define fuzzy sets extensively
  - Reasoning with fuzzy GCIs, allowing to constrain the truth value of a GCI

- **Reasoning preserving** procedure into a crisp KB.

- Alternative approach to achieve fuzzy ontologies, **reusing** currently existing crisp ontology languages and reasoners

- The semantics of fuzzy GCIs:
  - Allows fuzzy GCIs holding to some degree
  - Reduces the size of the resulting TBox w.r.t. Zadeh implication
  - However, it imposes some counter-intuitive effects
Future work

- **Empirical evaluation** to test the translation
- Consider different **fuzzy operators**
  - In particular, avoid the counter-intuitive effects of the Kleene-Dienes implication
- Include a crisp representation for **fuzzy datatypes**
  - Since OWL does not currently allow to define customised datatypes, consider OWL Eu
- Consider **more expressive DLs** and, in particular, **SROIQ**:
  - Subjacent DL of OWL 1.1
  - The additional expressivity in roles may help to overcome the asymmetry in fuzzy concept and role inclusion axioms
Questions?

Thank you very much for your attention