Tractable Reasoning Based on the Fuzzy $\mathcal{EL}^{++}$ Algorithm

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Abstract. Fuzzy Description Logics (f-DLs) are extensions of classic DLs that are capable of representing and reasoning about imprecise and vague knowledge. Though reasoning algorithms for very expressive fuzzy DLs have been explored, an open issue in the fuzzy DL community is the study of tractable systems. In this paper we introduce the fuzzy extension of $\mathcal{EL}^{++}$, we provide its syntax and semantics together with a reasoning algorithm for the fuzzy concept subsumption problem, in which other problems related to fuzzy DLs can be reduced.

1 Introduction

Fuzzy Description Logics (f-DLs) [5] are extensions of classic DLs capable of representing and reasoning about imprecise and vague knowledge. Following the progress in the classic DL community, reasoning algorithms for tractable fuzzy DLs have been explored. In [7] Straccia et al. introduced a fuzzy extension of the DL-Lite language while Pan et al. [3] presented the very first efficient and scalable system for f-DL-Lite which is able to answer expressive fuzzy conjunctive queries over millions of data. The current bibliography includes two fuzzy extensions of $\mathcal{EL}$. First Vojtáš presented a fuzzy extension of $\mathcal{EL}$ [8] which differs from most fuzzy DL languages because it interprets conjunction as a fuzzy aggregation rather than fuzzy intersection while in [4] Stoilos et al. examined a fuzzy extension of the tractable algorithm $\mathcal{EL}^{+}$.

In this paper we introduce the fuzzy extension of $\mathcal{EL}^{++}$. Similar to the fuzzy $\mathcal{EL}^{+}$ language, fuzzy $\mathcal{EL}^{++}$ allows for concept axioms with degrees of truth i.e. fuzzy subsumption axioms [6]. Furthermore it allows for nominals and the bottom concept increasing in that way its expressiveness compared to its previous extensions.

2 The Fuzzy $\mathcal{EL}^{++}$ Language

The structural elements of the fuzzy $\mathcal{EL}^{++}$ language are *concept names* N_C, *role names* N_R and *individuals* N_I. As usual individuals represent the objects of our universe, concept names represent fuzzy sets of individuals and role names represent binary fuzzy relationships between individuals. The semantics of fuzzy $\mathcal{EL}^{++}$ are given via a fuzzy interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ that is consisted of a domain $\Delta^{\mathcal{I}}$ which is a non empty set of individuals and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ which maps each $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, each $A \in N_C$ to a membership function $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$ and each $r \in N_R$ to a membership function $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$.

Fuzzy $\mathcal{EL}^{++}$ allows us to inductively define complex concept descriptions using the constructors shown in the table below, along with their semantics. Our language, similar to [4], allows for *fuzzy general concept inclusions* (*fuzzy GCIs*, first introduced in [6]) of the form $C \sqsubseteq^d D$ and *role inclusion axioms* (*RIs)* of the form $r_1 \circ \ldots \circ r_k \sqsubseteq s$. The semantics of fuzzy GCIs and RIs are given in the same table where the operator $\circ^t$ corresponds to the sup-t composition described in [4]. The set of fuzzy GCIs and RIs is called a *constraint box (CBox)* $\mathcal{C}$ (similar to [1]). An interpretation $\mathcal{I}$ is a model of a CBox $\mathcal{C}$ iff, for each GCI and RI in $\mathcal{C}$, the conditions described in the middle part of table are satisfied.

The fuzzy $\mathcal{EL}^{++}$ language also allows for an *assertional box (ABox)* $\mathcal{A}$ i.e. a finite set of *concept* and *role assertions* that are used to describe a snapshot of our world. The syntax along with the semantics, of concept and role assertions, is described in the table below. An interpretation $\mathcal{I}$ is a model of an ABox $\mathcal{A}$ iff, each concept and role assertion in $\mathcal{A}$ is satisfied.

Finally an interpretation $\mathcal{I}$ is a model of a fuzzy knowledge base $\mathcal{K} = \{\mathcal{A}, \mathcal{C}\}$ consisting of an ABox $\mathcal{A}$ and a CBox $\mathcal{C}$ iff it is, at the same time, a model of $\mathcal{A}$ and $\mathcal{C}$.

Name	Syntax	Semantics
top	$\top$	$\top^{\mathcal{I}}(x) = 1$
bottom	$\bot$	$\bot^{\mathcal{I}}(x) = 0$
nominal	$\{a\}$	$\{a\}^{\mathcal{I}}(x) = \begin{cases} 1 & \text{when } x = a^{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}$
conjunction	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}}(x) = min\left(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\right)$
existential restriction	$\exists r.C$	$(\exists r.C)^{\mathcal{I}}(x) = sup_{y \in \Delta^{\mathcal{I}}}\left(min\left(r^{\mathcal{I}}(x,y), C^{\mathcal{I}}(y)\right)\right)$
GCI	$C \sqsubseteq^d D$	$min\left(C^{\mathcal{I}}(x), d\right) \le D^{\mathcal{I}}(x)$
RI	$r_1 \circ \ldots \circ r_k \sqsubseteq s$	$\left[r_1^{\mathcal{I}} \circ^t \ldots \circ^t r_k^{\mathcal{I}}\right](x,y) \le s^{\mathcal{I}}(x,y)$
concept assertion	$C(a) \ge d$	$C^{\mathcal{I}}\left(a^{\mathcal{I}}\right) \ge d$
role assertion	$r(a,b) \ge d$	$r^{\mathcal{I}}\left(a^{\mathcal{I}}, b^{\mathcal{I}}\right) \ge d$

3 Deciding Subsumption in fuzzy $\mathcal{EL}^{++}$

The Fuzzy $\mathcal{EL}^{++}$ is an algorithm for deciding fuzzy concept subsumption. Following to [1] other problems can be reduced to the fuzzy concept subsumption problem. The proposed algorithm, similar to that presented in [1], demands for a normalized form of CBoxes. The normalization process operates similarly to that described in [1], having as main difference the use of fuzzy general concept inclusions instead of concept inclusions.

In order to decide for fuzzy subsumption between two concept names C and D w.r.t. a normalized CBox $\mathcal{C}$ i.e. $C \sqsubseteq^d_{\mathcal{C}} D$, it is sufficient to decide for fuzzy subsumption between a nominal $\{o\}$ and a concept B w.r.t. a CBox $\mathcal{C}' =$

$\mathcal{C} \cup \{\{o\} \sqsubseteq C, D \sqsubseteq B\}$, where o is a new individual name and B is a new concept name not appearing in $BC_{\mathcal{C}}$.

Let $R_{\mathcal{C}}$ denote the set of all role names in $\mathcal{C}$, where $\mathcal{C}$ is the normal form of the CBox to be classified. Our algorithm similar to [1,4] is based on two mappings, a mapping S from $BC_{\mathcal{C}} \times BC_{\mathcal{C}}$ to $[0,1]$ and a mapping R from $R_{\mathcal{C}} \times BC_{\mathcal{C}} \times BC_{\mathcal{C}}$ to $[0,1]$. Intuitively each of these two mappings has the purpose of making implicit fuzzy subsumption relationships, explicit as follows: $S(C,D) = d$ implies that $C \sqsubseteq^d_{\mathcal{C}} D$ and $R(r,C,D) = d$ implies that $C \sqsubseteq^d_{\mathcal{C}} \exists r.D$.

In the initialization of S we have that $S(C,D) := 1$ if $D = C$ or $D = \top$, otherwise $S(C,D) = 0$ for each $C, D \in BC_{\mathcal{C}} \cup \{\bot\}$. In the initialization of R we have that $R(r,C,D) = 0$ for each $r \in R_{\mathcal{C}}$, $C, D \in BC_{\mathcal{C}} \cup \{\bot\}$. After the initialization our algorithm proceeds with the application of the following completion rules, until no rule can be applied.

CR1	If $S(C,C') = d_1$, $C' \sqsubseteq^{d_2} D \in \mathcal{C}$ and $S(C,D) < min(d_1,d_2)$ then $S(C,D) = min(d_1,d_2)$
CR2	If $S(C,C_1) = d_1$, $S(C,C_2) = d_2$, $C_1 \sqcap C_2 \sqsubseteq^{d_3} D \in \mathcal{C}$, and $S(C,D) < min(d_1,d_2,d_3)$ then $S(C) := min(d_1,d_2,d_3)$
CR3	If $S(C,C') = d_1$, $C' \sqsubseteq^{d_2} \exists r.D \in \mathcal{C}$ and $R(r,C,D) < min(d_1,d_2)$ then $R(r,C,D) := min(d_1,d_2)$
CR4	If $R(r,C,D) = d_1$, $S(D,C') = d_2$, $\exists r.C' \sqsubseteq^{d_3} E \in \mathcal{C}$ and $S(C,E) < min(d_1,d_2,d_3)$ then $S(C,E) = min(d_1,d_2,d_3)$
CR5	If $R(r,C,D) > 0$, $S(D,\bot) > 0$ and $S(C,\bot) = 0$, then $S(C,\bot) = 1$
CR6	If $S(C,\{a\}) = 1$, $S(E,\{a\}) = 1$ and $C \leadsto_d E$, then for each $D \in BC_{\mathcal{C}}$, if $S(C,D) < min(d, S(E,D))$ $S(C,D) := min(d, S(E,D))$
CR7	If $R(r,C,D) = d$, $r \sqsubseteq s \in \mathcal{C}$ and $R(s,C,D) < d$ then $R(s,C,D) := d$
CR8	If $R(r_1,C,D) = d_1$, $R(r_2,D,E) = d_2$, $r_1 \circ r_2 \sqsubseteq r_3 \in \mathcal{C}$ and $R(r_3,C,E) < min(d_1,d_2)$ then R$(r_3,C,D) := min(d_1,d_2)$
CR9	If $S(C,\{a\}) > 0$ for some nominal $\{a\}$ and $S(C,\{a\}) < 1$ then $S(C,\{a\}) := 1$

Definition 1. *The abbreviation $\leadsto_d$ used in rule CR6 is similar to the abbreviation adopted in [1]. The relation $C \leadsto_d E$ between two concept names $C, E \in BC_{\mathcal{C}}$ indicates that there exists a set of concept names $C_1, \ldots, C_{k+1} \in BC_{\mathcal{C}}$ and role name $r_1, \ldots, r_k \in R_{\mathcal{C}}$, such that it holds that $min(R(r_1,C_1,C_2), \ldots, R(r_k,C_k,C_{k+1})) = d$ where $C_{k+1} = E$ and either $C_1 = C$ or $C_1 = \{a\}$, where $\{a\}$ is a nominal in $BC_{\mathcal{C}}$.*

Lemma 1. *Let S be the mapping obtained after the exhaustive application of rules for a normalized CBox $\mathcal{C}$ and let $\{o\}$ be a nominal and B be a concept name in $\mathcal{C}$. Then $\{o\} \sqsubseteq^d_{\mathcal{C}} B$ holds iff $S(\{o\},B) \geq d$ or* there is some nominal $\{a\} \in BC_{\mathcal{C}}$ such that $S(\{a\},\bot) > 0$.

Theorem 1. *The algorithm we have developed for fuzzy subsumption between a nominal and a concept is sound and complete and operates in polynomial time.*

4 Conclusions and Future Work

In this paper we have presented a fuzzy extension of the tractable DL language $\mathcal{EL}^{++}$, fuzzy-$\mathcal{EL}^{++}$. The main contributions of our algorithm compared to the one of fuzzy-$\mathcal{EL}^{+}$ is that we introduce nominals and the bottom concept. The introduction of nominals allows for reasoning w.r.t. some assertional knowledge in contrast to the fuzzy-$\mathcal{EL}^{+}$ language which only allowed for a CBox. Therefore the instance problem w.r.t. to some ABox and some CBox can be described and solved in fuzzy $\mathcal{EL}^{++}$. Additionally the presence of the bottom concept permits concept satisfiability and ABox consistency reasoning services. Finally the presence of the bottom concept allows to imply disjointness between concepts i.e. $C \sqcap D \sqsubseteq \perp$ and along with the existence of nominals allows to express unique name assumption between two individuals i.e. $\{a\} \sqcap \{b\} \sqsubseteq \perp$.

Further extensions of our language would be lead by the extensions of the corresponding crisp language. We could examine if an extension of our language with concrete domains is possible and the way in which this would affect its complexity. Furthermore we could also examine if our language could be extended, retaining its tractability, with the existence of domain and range properties restrictions similarly to the extension of the crisp algorithm presented in [2].

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