Deciding Fuzzy Description Logics by Type Elimination

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What is it all about?

A modern approach to find out what is in our paper¹:

⇒ Well, this all seems rather FUZZY!
Motivation

**Fuzzy Description Logics (FDLs):** extend crisp DLs semantically and syntactically to integrate **vagueness**

- A lot of concepts and relations in domain models are **not clearly defined** (e.g. `tall(·)`, `likes(·, ·)`)
- Useful: Observations about objects can be “contradicting” to some extent, e.g. something being hot and cold at the same time.

**Complexity of Reasoning in FDLs?**

- We could not find too many results, expect e.g. [Str01]
- **Intuitively:** extension of classical DLs implies “at least as hard as DLs”
- It is good to have a **variety of different tools** to tackle the problem!

**Reasoning in FDLs**

- Directly in the FDL world: mainly tableau-based methods, e.g. [Str01, SSSP06, SSP⁺07, SB07, LXLK06, HPS08]
- By translation to classical DLs, e.g. [Str04] (should always lead to a suboptimal solution!)
Research Question

Can we come up with a way to perform reasoning in FDLs that

- Solves a fundamental and useful reasoning problem
- Works directly in the level of FDLs (no translation to DLs)
- Works differently from tableau-based methods
- Can deal with GCIs?
What did we achieve?

- Designed a novel procedure \textbf{FixIt\textsubscript{ALC}} for deciding knowledge base (KB) satisfiability in the FDL \textsubscript{ALC}

- Formally proved soundness, completeness and termination of the algorithm and can show that the runtime behavior is worst-case optimal

- It is the first fixpoint-based decision procedure that has been proposed for FDL introducing a new class of inference procedures into FDL reasoning

- Our approach can deal with general terminologies (GCIs)
  - Together with [SSSP06, LXLK06, SB07, HPS08], one of the few possible approaches.
  - First non-tableau-based decision procedure to integrate GCIs
  - General terminologies are handled differently than in standard tableau-based method such as [SSSP06, LXLK06]
How did we achieve that?

- **FixIt**(\textit{ALC}) generalizes a type-elimination-based decision procedure [Pra80] for the (classical) modal logic \textit{K}, i.e. \textit{KBDD} [PSV06], to the FDL \textit{ALC}.

- Principle underlying \textit{KBDD} carries over to \textit{ALC}, but only in a different form than in [PSV06].

- Additionally we integrate (fuzzy) ABoxes and general TBoxes which are not dealt with in \textit{KBDD}.
### Syntax and Semantics of Concept Expressions

<table>
<thead>
<tr>
<th>Constructor</th>
<th>Syntax $E$</th>
<th>Semantics $E^I(o)$ (wrt. interp. $I$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>concept names</td>
<td>$A$</td>
<td>$A^I : \Delta^I \rightarrow [0, 1]$</td>
</tr>
<tr>
<td>role names</td>
<td>$R$</td>
<td>$R^I : \Delta^I \times \Delta^I \rightarrow [0, 1]$</td>
</tr>
<tr>
<td>universal truth / possibility</td>
<td>$\top$</td>
<td>$1$</td>
</tr>
<tr>
<td>concept conjunction</td>
<td>$C \sqcap D$</td>
<td>$\min(C^I(o), D^I(o))$</td>
</tr>
<tr>
<td>concept disjunction</td>
<td>$C \sqcup D$</td>
<td>$\max(C^I(o), D^I(o))$</td>
</tr>
<tr>
<td>concept negation</td>
<td>$\neg C$</td>
<td>$1 - C^I(o)$</td>
</tr>
<tr>
<td>univ. value restriction</td>
<td>$\forall R.C$</td>
<td>$\inf_{o' \in \Delta^I} {\max(1 - R^I(o, o'), C^I(o'))}$</td>
</tr>
<tr>
<td>ex. value restriction</td>
<td>$\exists R.C$</td>
<td>$\sup_{o' \in \Delta^I} {\min(R^I(o, o'), C^I(o'))}$</td>
</tr>
<tr>
<td>universal falsehood</td>
<td>$\bot$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
A (Minimalist) Fuzzy Description Logic: $\mathbf{ALC}$ [Str01]

Syntax and Semantics of Fuzzy Axioms

<table>
<thead>
<tr>
<th>Type</th>
<th>Axiom $\alpha$</th>
<th>Satisfaction of $\alpha$ by $\mathcal{I}$: $\mathcal{I} \models \alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{T}$: General Concept Inclusion (GCI)</td>
<td>$C \sqsubseteq D$</td>
<td>$C^\mathcal{I}(o) \leq D^\mathcal{I}(o)$ for all $o \in \Delta^\mathcal{I}$</td>
</tr>
<tr>
<td>$\mathcal{A}$: Fuzzy concept membership</td>
<td>$\langle i : C \bowtie d \rangle$</td>
<td>$C^\mathcal{I}(i^\mathcal{I}) \bowtie d$ for $\bowtie \in {\geq, \leq, =}$</td>
</tr>
<tr>
<td>$\mathcal{A}$: Fuzzy relation assertion</td>
<td>$\langle R(i, i') \geq d \rangle$</td>
<td>$R^\mathcal{I}(i^\mathcal{I}, i'^\mathcal{I}) \geq d$</td>
</tr>
</tbody>
</table>

Syntax and Semantics of a fuzzy KB

A **TBox** $\mathcal{T}$ is a finite set of GCIs. An **ABox** $\mathcal{A}$ is a finite set of fuzzy assertions. A **fuzzy knowledge base** $\mathcal{K}$ is a pair $\mathcal{K} = (\mathcal{T}, \mathcal{A})$

$\mathcal{I} \models \mathcal{T}$ iff. $\mathcal{I} \models \alpha$ for all $\alpha \in \mathcal{T}$

$\mathcal{I} \models \mathcal{A}$ iff. $\mathcal{I} \models \alpha$ for all $\alpha \in \mathcal{A}$

$\mathcal{I} \models \mathcal{K}$ iff. $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$
Reasoning in ALC

Given a fuzzy KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, fuzzy ABox axioms or GCIs $\alpha$, we can analyze particular semantic characteristics and interdependencies:

- $\mathcal{K}$ is **satisfiable** (or consistent) iff there is a model $\mathcal{I}$ for $\mathcal{K}$, i.e. an interpretation $\mathcal{I}$ such that

  $$\mathcal{I} \models \mathcal{T} \quad \text{and} \quad \mathcal{I} \models \mathcal{A}$$

- $\mathcal{K}$ **entails** $\alpha$ (denoted as $\mathcal{K} \models \alpha$) iff all models $\mathcal{I}$ of $\mathcal{K}$ satisfy $\alpha$

**Specific entailment problems:** concept equivalence, subsumption, disjointness, concept membership at least / most to a given degree, individual interrelation at least to a given degree, ...

**Complexity of checking KB satisfiability in ALC:**
ExpTime-complete [KH09, KH08]
Type Elimination: Overview

- Can be seen as a **model building procedure**

- Does not rely on systematic search in the first place, but instead constructs a canonical interpretation by means of a **fixpoint construction**

- The computed interpretation is in general **not tree-shaped** (as in tableau-based methods)
Preprocessing

Transform input KB $\mathcal{K}$ into a syntactically restricted normal form:

- **Normalize Fuzzy Axioms to the form $\langle \alpha \geq d \rangle$:**
  
  $\langle i : C \leq d \rangle \leadsto \langle i : \neg C \geq 1 - d \rangle$
  
  $\langle i : C = d \rangle \leadsto \langle i : C \geq d \rangle, \langle i : \neg C \geq 1 - d \rangle$

- **Expand all syntactic abbreviations:**
  
  $\exists R.D \leadsto \neg \forall R.\neg D$
  
  $\bot \leadsto \neg \top$

- **Convert all axioms into Box Normal Form (BNF):** Negation occurs only in front of concept names, $\top$ or expressions of the form $\exists R.D$:
  
  $\neg (C \sqcap D) \leadsto \neg C \sqcup \neg D$
  
  $\neg (C \sqcup D) \leadsto \neg C \sqcap \neg D$
  
  $\neg \neg C \leadsto C$
The closure of a normalized knowledge base $\text{cl}(\mathcal{K})$ is defined as the smallest set of concept expressions such that for all $C \in \text{sub}(\mathcal{K})$, if $C$ is not of the form $\neg D$, then $\{C, \neg C\} \subseteq \text{cl}(\mathcal{K})$

**In other words:** the closure of a KB contains for any concept expression occurring in KB the positive and negative form.

**How we use that:** The closure of a KB gives us the basic vocabulary to describe all relevant properties of individuals in interpretations.
Further, let $\text{PossDeg}(\mathcal{K})$ denote the set of all relevant possibility degrees that can be derived from $\mathcal{K}$ defined by

$$\text{PossDeg}(\mathcal{K}) = \{0, 0.5, 1\} \cup \{d|\langle \alpha \geq d \rangle \in \mathcal{A}\} \cup \{1 - d|\langle \alpha \geq d \rangle \in \mathcal{A}\}$$

[Str01, Str04] showed that if $\mathcal{K}$ is satisfiable, then there is as well a model of $\mathcal{K}$ which assigns possibility degrees in $\text{PossDeg}(\mathcal{K})$ only.

Hence, for our purposes we do not need to consider arbitrary possibility degrees $d \in [0, 1]$, but only the finite set $\text{PossDeg}(\mathcal{K})$ that can be derived from $\mathcal{K}$. 
We are living in a fuzzy world: Properties always hold to a certain degree

Definition (Fuzzy $\mathcal{K}$-Type)

A fuzzy $\mathcal{K}$-type $\tau$ is a maximal subset of $\text{cl}(\mathcal{K}) \times \text{PossDeg}(\mathcal{K})$ such that the following conditions are satisfied:

1. if $\langle C, d \rangle \in \tau$ and $\langle C, d' \rangle \in \tau$ then $d = d'$
2. if $C = \neg C'$ then $\langle C, d \rangle \in \tau$ iff $\langle C', 1 - d \rangle \in \tau$
3. if $C = C' \land C''$ then $\langle C, d \rangle \in \tau$ iff $\langle C', d' \rangle \in \tau$ and $\langle C'', d'' \rangle \in \tau$ and $d = \min(d', d'')$
4. if $C = C' \lor C''$ then $\langle C, d \rangle \in \tau$ iff $\langle C', d' \rangle \in \tau$ and $\langle C'', d'' \rangle \in \tau$ and $d = \max(d', d'')$
5. for all $C \sqsubseteq C' \in \mathcal{T}$: if $\langle C, d \rangle \in \tau$ and $\langle C', d' \rangle \in \tau$ then $d \leq d'$
6. if $C = \top$ then $\langle C, 1 \rangle \in \tau$. 
Fuzzy Types vs. Individuals

**Individuals**
- Basic elements to compose interpretations
- Are assigned elementary properties that can be observed (wrt. an interpretation)
- Are interrelated with other individuals (to a certain degree)

**Fuzzy Types:**
- (Propositionally consistent) syntactic view on the properties of a possible individual
- State what elementary and complex properties can be observed about
- Modal properties constrain the way the individual can be interrelated to other individuals

**Hence:** Types are syntactic correspondents to individuals (used in interpretations for $\mathcal{K}$)

$\langle C, d \rangle \in \tau \sim$ the individual represented by $\tau$ is a member of $C$ to degree $d$
Set of all Types = Vocabulary to construct any interpretation $\mathcal{I}$ of $\mathcal{K}$

- We simply need to fix how to interconnect the individuals they represent

Canonical Interconnection and Interpretation:

- Given a set $T$ of $\mathcal{K}$-types, interconnect them in a standard (or canonical way) $\Delta_R(\tau, \tau')$ (see our paper)
- The resulting canonical interpretation $\mathcal{I}(T)$ is almost directly a model of the input KB $\mathcal{K}$

$$C^{\mathcal{I}(T)}(\tau) = d \text{ iff } \langle C, d \rangle \in \tau$$

for almost all $C \in \text{cl}(\mathcal{K})$.

If (*) would be satisfied for all $C \in \text{cl}(\mathcal{K})$, then we would have $\mathcal{I}(T) \models C \sqsubseteq C'$ for all $C \sqsubseteq C' \in \mathcal{T}$ by clause (5) in our definition of $\mathcal{K}$-types, i.e. our canonical interpretation would be a model for $\mathcal{T}$. 
What is the Problem with the Canonical Interpretation?

We know that

\[ C^{\mathcal{I}(T)}(\tau) = d \text{ iff } \langle C, d \rangle \in \tau \quad (\ast) \]

for almost all \( C \in \text{cl}(\mathcal{K}) \).

That (\ast) is satisfied by \( \mathcal{I}(T) \) is straightforward for the cases of concept names \( C, \top \), or complex concepts of the form \( C = C' \cap C'' \), \( C = C' \cup C'' \), \( C = \neg C' \), as well as the \( C^{\mathcal{I}(T)}(\tau) \geq d \) case for \( C = \forall R.C \) by our definition of types and the definition of \( \Delta_R \).

The only cases where (\ast) can be violated by \( \mathcal{I}(T) \) is for types \( \tau \) containing universally role restricted concepts \( \forall R.C \) that are assigned a possibility degree which is too small (wrt. the \( R \)-successor types \( \tau' \) in \( \mathcal{I}(T) \)) to properly reflect the semantics of \( \forall R.C \) in \( \mathcal{ALC} \), i.e. to coincide with the greatest lower bound of the set

\[ \{ \max(1 - R^{\mathcal{I}(T)}(\tau, \tau'), C^{\mathcal{I}(T)}(\tau')) \mid \tau' \in T \} \]
How to fix this Problem?

Call any type \( \tau \) containing universally role restricted concepts \( \forall R.C \) that are assigned a possibility degree which is too small (wrt. the \( R \)-successor types \( \tau' \) in \( \mathcal{I}(T) \)) to properly reflect the semantics of \( \forall R.C \) in \( \mathcal{ALC} \), i.e. does not coincide with the greatest lower bound of the set

\[
\{ \max(1 - R^{\mathcal{I}(T)}(\tau, \tau'), C^{\mathcal{I}(T)}(\tau')) \mid \tau' \in T \}
\]

a bad type (wrt. the given set of types \( T \)).

Bad types \( \tau \in T \) can be detected easily: they satisfy that there exist \( R \in R \), \( C \in C(\Sigma) \), \( d \in \text{PossDeg}(\mathcal{K}) \) s.t. \( \langle \forall R.C, d \rangle \in \tau \) and for all \( \tau' \in T \): if \( \langle C, d' \rangle \in \tau' \) then \( \max(1 - \Delta_R(\tau, \tau'), d') > d \).

Basic Idea: Bad types are the only trouble that prevent us from ending up with a model for \( \mathcal{T} \). Remove (iteratively) all bad types from \( T \)!
Computation of a Canonical Model by Type Elimination
The Canonical Model is the “Maximal Model”

Lemma
Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be any model of $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. For each individual $o \in \Delta^\mathcal{I}$ we define its corresponding type

$$\tau(o) := \{\langle C, d \rangle \in \text{cl}(\mathcal{K}) \times \text{PossDeg}(\mathcal{K}) | C^\mathcal{I}(o) = d\}$$

Then, $\Delta_R(\tau(o), \tau(o')) \geq R^\mathcal{I}(o, o')$ for all $o, o' \in \Delta^\mathcal{I}$.

► Last step: Check an ABox model can be derived from $\mathcal{I}(T^*)$
Our Decision Procedure **FixIt**(\(\text{ALC}\))

**procedure** satisfiable\((\mathcal{K})\): boolean

\(T := \{ \tau \mid \tau \text{ is a } \mathcal{K}-\text{type} \};\)

**repeat**

\[
T' := T;
T := T' \setminus \text{badtypes}(T');
\]**until** \(T = T'\);

**if** there exists a total function \(\pi : \text{Ind}_A \rightarrow T\) s.t. \(\langle C, d' \rangle \in \pi(o)\) and \(d \leq d'\) for each \(\langle o : C \geq d \rangle \in A\), and \(\Delta_R(\pi(o), \pi(o')) \geq d\) for each \(\langle R(o, o') \geq d \rangle \in A\) **then**

**return** true;

**end**

**return** false;

**function** badtypes\((T)\) : \(2^T\)

**return** \(\{ \tau \in T \mid \langle \forall R.C, d \rangle \in \tau\) and for all \(\tau' \in T\): if \(\langle C, d' \rangle \in \tau'\) then \(\max(1 - \Delta_R(\tau, \tau'), d') > d\}\);
Future Work

- Study means of **implicit representation of sets of fuzzy types** known from Symbolic Model Checking [McM93], in particular OBDDs.
- A major question concerning optimization: **how to implement the final test of the algorithm efficiently**, e.g. by heuristic search using the information in the ABox effectively to find the required mapping.
- The **integration of optimizations** such as full vs. lean representations or particle vs. types as discussed in [PSV06].
- **Evaluate the efficiency** of the method by an implementation and comparison to tableau-based systems for FDLs.
- **Study the bottom-up variant** of KBDD in the context of FDLs too, check if the integration of ABoxes can be done more efficiently in such a variant.
- **Investigate** to what extend the method can cover other semantics for FDLs (e.g. other t-norms) and extended constructs, such as fuzzy modifiers and concrete domains.
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Umberto Straccia.
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Umberto Straccia.
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