Axiomatic First-Order Probability for the Semantic Web

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Uncertainty is ubiquitous

- “Prediction is difficult, especially the future.” – Yogi Berra
- In an open world, attempts to nail down an unambiguous meaning and definite truth-value for every statement are doomed to failure.

Probability formalizes reasoning under uncertainty

- “Symbolic logic is a model [of deductive thought] in much the same way that modern probability theory is a model for situations involving chance and uncertainty.” – Enderton (2001)
- Probability allows us to draw useful conclusions when our knowledge falls short of certainty

There is vigorous debate over:

- Semantics of probability
- How to combine probability with classical logic
Mathematical Probability

- Probabilities are assigned to *events*
  - Event represents uncertain outcome
  - Mathematically, events are subsets of a *sample space* \( \Omega \)
  - (For uncountable \( \Omega \), we must restrict events to *measurable* subsets of \( \Omega \))

- A *probability measure* \( P(\cdot) \) satisfies the following axioms:
  - \( P(A) \geq 0 \) for all measurable events \( A \)
  - \( P(\Omega) = 1 \)
  - If \( A_1, A_2, \ldots \) is a sequence of measurable events such that \( A_i \cap A_j = \emptyset \) then \( P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots \)

- The conditional probability of \( A \) given \( B \)
  for any two events \( A \) and \( B \) is defined as
  - a number \( P(A|B) \) satisfying:
    - \( P(A|B)P(B) = P(A \cap B) \)
Propositional Logic and Probability

- There is a natural way to define probabilities in a propositional language with finitely many sentence symbols.
- Each sentence symbol specifies an event:
  - Event $A$ corresponding to sentence symbol $Q$ occurs if and only if $Q$ is true.
  - $Q_1 \lor \cdots \lor Q_n$ corresponds to $A_1 \cup \cdots \cup A_n$.
  - $Q_1 \land \cdots \land Q_n$ corresponds to $A_1 \cap \cdots \cap A_n$.
  - Similarly for the other logical connectives.
- We can define a probability measure over truth values of the $Q_i$:
  - Probability measure on truth values of sentence symbols gives rise to probability for each wff.
  - Probability measure can be defined consistently and parsimoniously using conditional independence.
- This idea can be extended to languages with infinitely many sentence symbols.
Second-order Logic
- Specify probability density functions directly
- Represent probabilities with real numbers
  - No completeness or compactness theorem (valid sentence may not be provable)
  - Even logicians do not agree on semantics (or on whether it is meaningful to quantify over all functions and relations)

First-order Logic
- Cannot refer directly to properties, functions or sentences
- Represent arbitrarily fine-grained degrees of plausibility
- Completeness and compactness theorem (valid sentences are provable)
- Well-understood and universally accepted semantics
- Can refer indirectly to sentences (via Gödel numbers)
- Most SW languages are based on a subset of first-order logic

Second-order logic with general semantics
- Second-order syntax with first-order model theory
- Talk about sentences and retain other benefits of FOL
Axiomatic First-Order Theory

- Represent knowledge explicitly as finite computational structure
- Contradictions and entailments can be detected in finite time
- Consequences are effectively enumerable
- *Can we formalize probability as a first-order axiomatic theory?*
No-Go Results (i)

- Gaifman (1964) assigned probabilities to sentences of a first-order language
  - Extend original language $\mathcal{L}$ to a new language $\mathcal{L}^*$ with additional individual constants to cover all objects in the domain
  - Assign probability measure to quantifier-free sentences of $\mathcal{L}^*$
  - Extend to probability measure on all sentences via Gaifman’s condition:
    \[ P(\forall x \psi(x)) \text{ is supremum of } P(\psi(\kappa_1) \lor \cdots \lor \psi(\kappa_n)), \text{ for all finite conjunctions } \psi(\kappa_1) \lor \cdots \lor \psi(\kappa_n) \text{ of sentences, formed by substituting constant terms of the extended language } \mathcal{L}^* \text{ into } \psi(x) \]
  - *Measure-model* semantics defines a probability measure on possible worlds

- Gaifman and Snir (1982) studied definability of probabilities and tests for satisfaction
  - Refer to sentences indirectly via Gödel numbers
  - Semantics restricts mathematical sublanguage to intended interpretation on natural numbers; therefore:
    - Probabilities are not definable on mathematical sublanguage
    - All definable probability functions on empirical sublanguage are “dogmatic” (assign probability zero to some satisfiable sentence)
No-Go Results (ii)

- Bacchus criticized Gaifman’s approach because it “fail[s] to address some of the main concerns of AI”
  - Cannot represent assertions about probabilities, e.g.:
    - “The false positive probability is less than 0.05”
    - “Rain is more likely today than it was yesterday.”
- Abadi and Halpern (1994) examined first-order logics that can reason both with and about probability
  - “…first-order …language for reasoning about probabilities ought to have easily comprehensible syntax and semantics.
  - “Ideally, the validity problem would not be worse than for first-order logic, and we would have a complete axiomatization…”
  - But “…as long as [the language] is sufficiently rich, the validity problem for first-order reasoning about probability is wildly undecidable.”
  - No complete axiomatization is possible
- Undecidability results apply even if probabilities are restricted to rational numbers
Addressing the Roadblocks

- Probabilities are usually formalized as real numbers, and real numbers cannot be axiomatized in a first-order theory
  - Real numbers = ordered field + least upper bound axiom
  - Least upper bound axiom refers to all bounded subsets of the real numbers
  - We can formulate a first-order least upper bound axiom that applies to all definable bounded subsets
  - This is the theory of real closed fields

- FOL cannot refer to sentences
  - We can refer indirectly to sentences via their Gödel numbers

- We cannot define a “truth function” on the natural numbers
  - Any definable first-order probability function must be uncertain about some statements about the natural numbers
Desirable Features of Probability Logic

- Express statements about domain and about probabilities
- Express arbitrarily fine-grained degrees of likelihood
- Define a probability for every sentence in the language
- Define non-dogmatic distributions
- Condition explicitly on all background knowledge (mathematical, logical, domain)
- Discover any contradiction in finite time
- Support learning from observation
- Deal appropriately with infinite limits

All these can be achieved by formalizing probability as an axiomatic first-order theory
**The Axioms: Real Closed Field**

**R1:** *Additive and multiplicative closure:* For all \(x\) and \(y\), \(\mathcal{R}(x)\) and \(\mathcal{R}(y)\) imply \(\mathcal{R}(x+y)\) and \(\mathcal{R}(x \cdot y)\).

**R2:** *Commutativity of addition and multiplication:* For all \(x\) and \(y\), \(\mathcal{R}(x)\) and \(\mathcal{R}(y)\) imply \(x+y = y+x\) and \(x \cdot y = y \cdot x\).

**R3:** *Associativity of addition and multiplication:* For all \(x\), \(y\), and \(z\), \(\mathcal{R}(x)\), \(\mathcal{R}(y)\) and \(\mathcal{R}(z)\) imply \((x+y) + z = x + (y+z)\) and \((x \cdot y) \cdot z = x \cdot (y \cdot z)\).

**R4:** *Additive and multiplicative identity:* \(\mathcal{R}(0)\) and \(\mathcal{R}(1)\) and \(0 \neq 1\) and for all \(x\), \(\mathcal{R}(x)\) implies \(x+0 = x\) and \(x \cdot 1 = x\).

**R5:** *Additive and multiplicative inverses:* For all \(x\), \(\mathcal{R}(x)\) implies there exists \(y\) such that \(x + y = 0\). For all \(x\), \(\mathcal{R}(x)\) and \(x \neq 0\) implies there exists \(z\) such that \(xz = 1\).

**R6:** *Distributivity of multiplication over addition:* For all \(x\), \(y\), and \(z\), \(\mathcal{R}(x)\), \(\mathcal{R}(y)\) and \(\mathcal{R}(z)\) imply \(x \cdot (y+z) = (x \cdot y) + (x \cdot z)\).

**R7:** *Total ordering:* For all \(x\), \(y\), and \(z\), \(\mathcal{R}(x)\), \(\mathcal{R}(y)\) and \(\mathcal{R}(z)\) imply \((x \leq y \text{ or } y \leq x)\) and (if \(x \leq y\) and \(y \leq x\) then \(x = y\)) and (if \(x \leq y\) and \(y \leq z\) then \(x \leq z\)).

**R8:** *Agreement of ordering with field operations:* For all \(x\), \(y\), and \(z\), \(\mathcal{R}(x)\), \(\mathcal{R}(y)\) and \(\mathcal{R}(z)\) imply if \((x \leq y\) then \(x+z \leq y+z\)) and (if \(0 \leq x\) and \(0 \leq y\) then \(0 \leq x \cdot y\)).

**R9:** *First-order closure:* The following axiom schema holds for all one-place formulas \(q(x)\):

\[
\forall x \ (q(x) \to \mathcal{R}(x)) \land \exists x \ q(x) \land \exists y \ (\mathcal{R}(y) \land \forall x \ (q(x) \to x \leq y)) \to \\
\exists y \ (\mathcal{R}(y) \land \forall x \ (q(x) \to x \leq y) \land \forall z \ (\mathcal{R}(z) \land \forall x \ (q(x) \to x \leq z) \iff y \leq z)).
\]
The Axioms: Natural Numbers

**Integer arithmetic.** The following axioms, together with the real closed field axioms, provide enough power for Gödel numbering and reasoning about provability:

N1. \( \forall x \, \mathcal{N}(x) \rightarrow \mathcal{R}(x) \)

N2. \( \mathcal{N}(0) \)

N3. \( \forall x \, \mathcal{N}(x) \rightarrow \mathcal{N}(x+1) \)

N4. \( \forall x \, \forall y \, \mathcal{N}(x) \land \mathcal{N}(y) \rightarrow ((x < y+1) \rightarrow (x \leq y)) \)

N5. \( \forall x \, \mathcal{N}(x) \rightarrow \neg (x < 0) \)

N6. Induction axiom schema: all universal closures of formulas

\[ \mathcal{N}(x) \rightarrow (\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x+1))) \rightarrow \forall x \, \varphi(x) \]

where \( \varphi(x) \) has \( x \) (and possibly other variables) free.
The Axioms: Probability

- Probability axioms are stated informally but can be formalized as first-order axioms.
- Refer to sentence $\sigma$ indirectly through its Gödel number $\#\sigma$.
- $\mathcal{P}(\#\sigma, \#\varphi(x))$ represents probability of $\sigma$ given sentences represented by formula $\varphi(x)$.
- Suppress Gödel numbers for readability, e.g., $\mathcal{P}(\sigma \mid \varphi)$.
- $A^*$ represents mathematical and domain axioms.
- Probability axioms are universally quantified over (Gödel numbers of) sentences $\sigma$ and $\tau$, and formulas $\varphi$.

**P1.** $0 \leq \mathcal{P}(\sigma \mid \varphi) \leq 1$.

**P2.** If $A^* \vdash \sigma$, then $\mathcal{P}(\sigma \mid A^*) = 1$.

**P3.** If $\mathcal{P}(\sigma \land \tau \mid \varphi) = 0$, then $\mathcal{P}(\sigma \lor \tau \mid \varphi) = \mathcal{P}(\sigma \mid \varphi) + \mathcal{P}(\tau \mid \varphi)$.

**P4.** $\mathcal{P}(\sigma \land \tau \mid \varphi) = \mathcal{P}(\sigma \mid \tau, \varphi) \times \mathcal{P}(\tau \mid \varphi)$

**P5.** If $\sigma \leftrightarrow \tau$, then $\mathcal{P}(\sigma \mid \varphi) = \mathcal{P}(\tau \mid \varphi)$, and $\mathcal{P}(\gamma \mid \sigma, \varphi) = \mathcal{P}(\gamma \mid \tau, \varphi)$ for all sentences $\gamma$.

**P6.** $\mathcal{P}(\forall x \ \psi(x) \mid \varphi)$ is equal to the supremum of the values $\mathcal{P}(\psi(\kappa_1) \lor \cdots \lor \psi(\kappa_n) \mid \varphi)$, for all finite conjunctions $\psi(\kappa_1) \lor \cdots \lor \psi(\kappa_n)$ of sentences, formed by substituting constant terms of $\mathcal{C}^*$ into $\psi(x)$. 
Semantics

- Standard first-order model theoretic semantics applies.
- A model (or possible world) consists of:
  - A domain $D$
  - A function on $D^n$ for each $n$-ary function symbol
  - A subset of $D^n$ for each $n$-ary predicate symbol
  - An element of $D$ for each constant symbol

such that every axiom of $A^*$ is true in the model.

- Certainty restriction: Without affecting any probabilities, we can add an axiom schema concluding the negation of a sentence that provably has probability zero.

- Measure models. If a probability is defined for every sentence then there is a unique measure model; otherwise there is a set of measure models.
…for the Semantic Web

- First-order languages provide well-known advantages, e.g.
  - Explicit finite computational representation
  - Complete proof theory
  - Compatibility with SW languages
- Can translate to second-order language with general semantics
- Provides unified semantics for a variety of probability languages making different expressivity / tractability tradeoffs
Thank You!