Finite Fuzzy Description Logics: A Crisp Representation for Finite Fuzzy $\mathcal{ALCH}$

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Abstract. Fuzzy Description Logics (DLs) are a formalism for the representation of structured knowledge affected by imprecision or vagueness. In the setting of fuzzy DLs, restricting to a finite set of degrees of truth has proved to be useful. In this paper, we propose finite fuzzy DLs as a generalization of existing approaches. We assume a finite totally ordered set of linguistic terms or labels, which is very useful in practice since expert knowledge is usually expressed using linguistic terms. Then, we consider any smooth t-norm defined over this set of degrees of truth. In particular, we focus on the finite fuzzy DL $\mathcal{ALCH}$, studying some logical properties, and showing the decidability of the logic by presenting a reasoning preserving reduction to the non-fuzzy case.

1 Introduction

It has been widely pointed out that classical ontologies are not appropriate to deal with imprecise and vague knowledge, which is inherent to several real-world domains. Since fuzzy logic is a suitable formalism to handle these types of knowledge, there has been an important interest in generalize the formalism of Description Logics (DLs) [1] to the fuzzy case [2].

It is well known that different families of fuzzy operators (or fuzzy logics) lead to fuzzy DLs with different properties [2]. For example, Gödel and Zadeh fuzzy logics have an idempotent conjunction, whereas Łukasiewicz and Product fuzzy logic do not. Clearly, different applications may need different fuzzy logics.

In fuzzy DLs, some fuzzy operators imply logical properties which are usually undesired. For instance, in Zadeh fuzzy logic concepts and roles do not fully subsume themselves [3]. Furthermore, Łukasiewicz logic may not be suitable for combining information, as the conjunction easily collapses to zero [4]. Hence, the study of new fuzzy operators is an interesting topic.

Assuming a finite set of degrees of truth is useful in the setting of fuzzy DLs, [3,5,6]. In the Zadeh case it is interesting for computational reasons [3]. In Gödel logic, it is necessary to show that the logic verifies the Witnessed Model Property [7]. In Łukasiewicz logic, it is necessary to obtain a non-fuzzy representation of the fuzzy ontology [6]. A question that immediately arise is whether this assumption is possible when different fuzzy logics are considered.

There is a recent promising line of research that tries to fill the gap between mathematical fuzzy logic and fuzzy DLs [7,8,9]. Following this path, we build on the previous research on finite fuzzy logics [10,11,12] and propose a generalization of the different fuzzy DLs under finite degrees of truth that have been proposed, as we consider any smooth t-norm defined over a chain of degrees of truth.

Instead of dealing with degrees of truth in $[0, 1]$, as usual in fuzzy DLs, we will assume a finite (totally ordered) set of linguistic terms or labels. For instance, $\mathcal{N} = \{\texttt{false}, \texttt{closeToFalse}, \texttt{neutral}, \texttt{closeToTrue}, \texttt{true}\}$. This makes possible to abstract from the numerical interpretations of these labels.

The use of linguistic labels as degrees in fuzzy DLs has already been proposed. U. Straccia proposed to take the degrees from an uncertainty lattice [13]. To guarantee soundness and completeness of the reasoning, the set of labels is assume to be finite. A recent extension of this work by other authors considers Zadeh $\mathcal{SHIN}$ [14]. Nowadays, finite chains are receiving more attention, since they are one of the building blocks of the first order t-norm based logic $L^{*}_{\sim}(\mathbf{S})\forall$, which can be used to define several related fuzzy DLs [8,9].

The benefits of this paper are two-fold: firstly, since experts' knowledge is usually expressed using a set of linguistic terms [11], the process of knowledge acquisition is easier. Secondly, we make possible to use new fuzzy operators in the setting of fuzzy DLs for the first time.

The remainder is organized as follows. Section 2 includes some preliminaries on finite fuzzy logics. Then, Section 3 defines a fuzzy extension of the DL $\mathcal{ALCH}$ based on finite fuzzy logics and discusses some logical properties. Section 4 shows the decidability of the logic by providing a reduction of fuzzy $\mathcal{ALCH}$ into crisp $\mathcal{ALCH}$. Finally, Section 5 sets out some conclusions and ideas for future research.

2 Finite Fuzzy Logics

Fuzzy set theory and fuzzy logic were proposed by L. Zadeh [15] to manage imprecise and vague knowledge. Here, statements are not either true or false, but they are a matter of degree.

Let X be a set of elements called the reference set, and let $\mathcal{S}$ be a totally ordered scale with e as minimum element and u as maximum. A *fuzzy subset* A of X is defined by a membership function $A(x) : X \rightarrow \mathcal{S}$ which assigns any $x \in X$ to a value in $\mathcal{S}$. Similarly as in the classical case, e means no-membership and u full membership, but now a value between them represents to which extent x can be considered as an element of X.

All crisp set operations are extended to fuzzy sets. The intersection, union, complement and implication are performed by a t-norm function, a t-conorm function, a negation function, and an implication function, respectively.

In the following, we consider finite chains of degrees of truth [10,11,12]. A *finite chain* of degrees of truth is a totally ordered set $\mathcal{N} = \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_p = 1\}$, where $p \geq 1$. For our purposes all finite chains with the same number of elements are equivalent. $\mathcal{N}$ can be understood as a set of linguistic terms or labels. For example, $\{\texttt{false}, \texttt{closeToFalse}, \texttt{neutral}, \texttt{closeToTrue}, \texttt{true}\}$.

Table 1. Popular fuzzy logics over a finite chain

Family	$\gamma_i \otimes \gamma_j$	$\gamma_i \oplus \gamma_j$	$\ominus\gamma_i$	$\gamma_i \Rightarrow \gamma_j$
Zadeh	$\min\{\gamma_i, \gamma_j\}$	$\max\{\gamma_i, \gamma_j\}$	γ_{p-i}	$\max\{\gamma_{p-i}, \gamma_j\}$
Gödel	$\min\{\gamma_i, \gamma_j\}$	$\max\{\gamma_i, \gamma_j\}$	$\begin{cases} \gamma_p, \ \gamma_i = 0 \\ \gamma_0, \ \gamma_i > 0 \end{cases}$	$\begin{cases} \gamma_p, \ \gamma_i \leq \gamma_j \\ \gamma_j, \ \gamma_i > \gamma_j \end{cases}$
Łukasiewicz	$\gamma_{\max\{i+j-p,0\}}$	$\gamma_{\min\{i+j,p\}}$	γ_{p-i}	$\gamma_{\min\{p-i+j,p\}}$

In the rest of the paper, we will use the following notion: $\mathcal{N}^+ = \mathcal{N} \setminus \{\gamma_0\}$, $+\gamma_i = \gamma_{i+1}$, $-\gamma_i = \gamma_{i-1}$. Let us also denote by $[\gamma_i, \gamma_j]$ the finite chain given by the subinterval of all $\gamma_k \in \mathcal{N}$ such that $i \leq k \leq j$.

T-norms, t-conorms, negations and implications can be restricted to finite chains. Table 1 shows some popular examples: Zadeh, Gödel, and Łukasiewicz.

The *smoothness condition* is a discrete counterpart of continuity on $[0, 1]$. A function $f : \mathcal{N} \to \mathcal{N}$ is *smooth* iff it satisfies the following condition for all $i \in \mathcal{N}^+$ $f(\gamma_i) = \gamma_j$ implies that $f(\gamma_{i-1}) = \gamma_k$ with $j - 1 \leq k \leq j + 1$. A binary operator is smooth when it is smooth in each place.

A *t-norm* on $\mathcal{N}$ is a function $\otimes : \mathcal{N}^2 \to \mathcal{N}$ satisfying commutativity, associativity, monotonicity, and some boundary conditions. Smoothness for t-norms is equivalent to the divisibility condition in $[0, 1]$, i.e., $\gamma_i \leq \gamma_j$ if and only if there exists $\gamma_k \in \mathcal{N}$ such that $\gamma_j \otimes \gamma_k = \gamma_i$. A t-norm $\otimes$ is *Archimedean* iff $\forall \gamma_1, \gamma_2 \in \mathcal{N} \setminus \{\gamma_0, \gamma_p\}$ there is $n \in \mathbb{N}$ such that $\gamma_1 \otimes \gamma_1 \cdots \otimes \gamma_1$ (n times) $< \gamma_2$.

Proposition 1. *There is one and only one Archimedean smooth t-norm on* $\mathcal{N}$ *given by* $\gamma_i \otimes \gamma_j = \gamma_{\max\{0,i+j-p\}}$. *Moreover, given any subset* J *of* $\mathcal{N}$ *containing* γ_0, γ_p, *there is one and only one smooth t-norm* $\otimes^J$ *on* $\mathcal{N}$ *that has* J *as the set of idempotent elements. In fact, if* J *is the set* $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$ *such a t-norm is given by:*

$$\gamma_i \otimes^J \gamma_j = \begin{cases} \gamma_{\max\{i_k, i+j-i_{k+1}\}} & \text{if } \gamma_i, \gamma_j \in [i_k, i_{k+1}] \text{ for some } 0 \leq k \leq m-1 \\ \gamma_{\min\{i,j\}} & \text{otherwise} \end{cases}$$

Note that the Archimedean smooth t-norm happens with $J = \{\gamma_0, \gamma_p\}$, and that the minimum happens with $J = \mathcal{N}$. It is worth to note that, as a consequence of Proposition 1, a finite smooth product t-norm is not possible.

Example 1. Given the finite chain $\mathcal{N} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ and the set $J = \{\gamma_0, \gamma_3, \gamma_5\}$, $\otimes^J$ is defined as:

	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5
γ_0	γ_0	γ_0	γ_0	γ_0	γ_0	γ_0
γ_1	γ_0	γ_0	γ_0	γ_1	γ_1	γ_1
γ_2	γ_0	γ_0	γ_1	γ_2	γ_2	γ_2
γ_3	γ_0	γ_1	γ_2	γ_3	γ_3	γ_3
γ_4	γ_0	γ_1	γ_2	γ_3	γ_3	γ_4
γ_5	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5

A negation function $\ominus$ on $\mathcal{N}$ is *strong* if it verifies $\ominus(\ominus\gamma) = \gamma, \forall \gamma \in \mathcal{N}$. There is only one strong negation on $\mathcal{N}$ and it is given by $\ominus\gamma_i = \gamma_{p-i}$

Given a smooth t-norm $\otimes$ and the strong negation $\ominus$, we can define the *dual* t-conorm $\oplus_\otimes$, as the function satisfying $\gamma_i \oplus_\otimes \gamma_j = \ominus((\ominus\gamma_i) \otimes (\ominus\gamma_j))$.

Proposition 2. *There is one and only one Archimedean smooth t-conorm on* $\mathcal{N}$ *given by* $\gamma_i \oplus \gamma_j = \gamma_{\min\{p,i+j\}}$. *Moreover, given any subset* J *of* $\mathcal{N}$ *containing* γ_0, γ_p, *there is one and only one smooth t-conorm* $\oplus^J$ *on* $\mathcal{N}$ *that has* J *as the set of idempotent elements. In fact, if* J *is the set* $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$ *such a t-conorm is given by:*

$$\gamma_i \oplus^J \gamma_j = \begin{cases} \gamma_{\min\{i_{k+1}, i+j-i_k\}} & \text{if } \gamma_i, \gamma_j \in [i_k, i_{k+1}] \text{ for some } 0 \leq k \leq m-1 \\ \gamma_{\max\{i,j\}} & \text{otherwise} \end{cases}$$

Note that the Archimedean smooth t-conorm happens with $J = \{\gamma_0, \gamma_p\}$, and that the maximum happens with $J = \mathcal{N}$.

A binary operator $\Rightarrow: \mathcal{N}^2 \to \mathcal{N}$ is said to be an *implication*, if it is non-increasing in the first place, non-decreasing in the second place, and satisfies some boundary conditions.

Given a smooth t-norm $\otimes$ and the strong negation $\ominus$, an *S-implication* $\Rightarrow_{s\otimes}$ is the function satisfying $\gamma_i \Rightarrow_{s\otimes} \gamma_j = \ominus(\gamma_i \otimes (\ominus\gamma_j)) = (\ominus\gamma_i) \oplus \gamma_j$.

Proposition 3. *Let* $\otimes^J : \mathcal{N}^2 \to \mathcal{N}$ *be a smooth t-norm with* $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$. *Then, the implication* $\Rightarrow_{s\otimes}$ *is given by:*

$$\gamma_i \Rightarrow_{s\otimes} \gamma_j = \begin{cases} \gamma_{\min\{p-i_k, i_{k+1}+j-i\}} & \text{if } \exists \gamma_{i_k} \in J \text{ such that } \ \gamma_{i_k} \leq \gamma_i, \gamma_{p-j} \leq \gamma_{i_{k+1}} \\ \gamma_{\max\{p-i,j\}} & \text{otherwise} \end{cases}$$

The Kleene-Dienes implication happens with the minimum t-norm, and the Łukasiewicz implication happens with the Archimedean t-norm.

Given a smooth t-norm $\otimes$, an *R-implication* $\Rightarrow_{r\otimes}$ can be defined as $\gamma_i \Rightarrow_{r\otimes} \gamma_j = \max\{\gamma_k \in \mathcal{N} | (\gamma_i \otimes \gamma_k) \leq \gamma_j\}$, for all $\gamma_i, \gamma_j \in \mathcal{N}$.

Proposition 4. *Let* $\otimes^J : \mathcal{N}^2 \to \mathcal{N}$ *be a smooth t-norm with* $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$. *Then, the implication* $\Rightarrow_{r\otimes}$ *is given by:*

$$\gamma_i \Rightarrow_{r\otimes} \gamma_j = \begin{cases} \gamma_p & \text{if } \gamma_i \leq \gamma_j \\ \gamma_{i_{k+1}+j-i} & \text{if } \exists \gamma_{i_k} \in J \text{ such that } \ \gamma_{i_k} \leq \gamma_j < \gamma_i \leq \gamma_{i_{k+1}} \\ \gamma_j & \text{otherwise} \end{cases}$$

Example 2. Given the t-norm in Example 1, $\Rightarrow_{r\otimes}$ is defined as follows, where the first column is the antecedent and the first row is the consequent:

	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5
γ_0	γ_5	γ_5	γ_5	γ_5	γ_5	γ_5
γ_1	γ_2	γ_5	γ_5	γ_5	γ_5	γ_5
γ_2	γ_1	γ_2	γ_5	γ_5	γ_5	γ_5
γ_3	γ_0	γ_1	γ_2	γ_5	γ_5	γ_5
γ_4	γ_0	γ_1	γ_2	γ_4	γ_5	γ_5
γ_5	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5

Gödel implication happens with the minimum t-norm, and the Łukasiewicz implication happens with the Archimedean t-norm.

A *QL-implication* is an implication verifying $\gamma_i \Rightarrow \gamma_j = (\ominus\gamma_i) \oplus (\gamma_i \otimes \gamma_j)$.

Proposition 5. *Let* $\otimes : \mathcal{N}^2 \to \mathcal{N}$ *be a smooth t-norm. The operator* $\gamma_i \Rightarrow \gamma_j = (\ominus\gamma_i) \oplus (\gamma_i \otimes \gamma_j)$ *is a QL-implication iff* $\oplus$ *is the Archimedean smooth t-conorm. Moreover, in this case,* $\gamma_i \Rightarrow_{ql\otimes} \gamma_j = \gamma_{p-i+z}$ *for all* $\gamma_i, \gamma_j \in \mathcal{N}$*, where* $\gamma_z = \gamma_i \otimes \gamma_j$*.*

Proposition 6. *Let* $\otimes^J : \mathcal{N} \times^J \mathcal{N} \to \mathcal{N}$ *be a smooth t-norm with* $J = \{0 = \gamma_{i_0} < \gamma_{i_1} < \cdots < \gamma_{i_{m-1}} < \gamma_{i_m} = 1\}$*. Then, the implication* $\Rightarrow_{ql\otimes}$ *is given by:*

$$\gamma_i \Rightarrow_{ql\otimes} \gamma_j = \begin{cases} \gamma_{\max\{p-i+i_k, p+j-i_{k+1}\}} & \text{if } \gamma_i, \gamma_j \in [i_k, i_{k+1}] \text{ for some } 0 \leq k \leq m-1 \\ \gamma_{p-i+j} & \text{if } \gamma_j \leq i_k \leq \gamma_i \text{ for some } i_k \in J \\ \gamma_p & \text{otherwise} \end{cases}$$

The Łukasiewicz implication happens with the minimum t-norm, and the KleeneDienes implication happens with the Archimedean t-norm (note the difference with respect to S-implications).

Interestingly, $\Rightarrow_{s\otimes}$ and $\Rightarrow_{ql\otimes}$ are smooth if and only if so is $\otimes$, but the smoothness condition is not preserved in general for R-implications.

Finally, we can also define D-implications. The name is due to the equivalence to the Dishkant arrow in orthomodular lattices. Note that D-implication are sometimes called NQL-implication. A *D-implication* is an implication satisfying $\gamma_i \Rightarrow \gamma_j = ((\ominus\gamma_i) \otimes (\ominus\gamma_j)) \oplus \gamma_j$ for all $\gamma_i, \gamma_j \in \mathcal{N}$. However, QL-implications and D-implications on $\mathcal{N}$ actually coincide. Given a set J and $\bar{J} = \{\gamma_{p-x} | \gamma_x \in J\}$, then $\Rightarrow_{ql\otimes^J}$ is equivalent to $\Rightarrow_{d\otimes^{\bar{J}}}$.

The notions of fuzzy relation, inverse relation, composition of relations, reflexivity, symmetry and transitivity can trivially be restricted to $\mathcal{N}$.

3 Finite Fuzzy $\mathcal{ALCH}$

In this section we define fuzzy $\mathcal{ALCH}$, a fuzzy extension of $\mathcal{ALCH}$ where:

- Concepts denote fuzzy sets of individuals.
- Roles denote fuzzy binary relations.
- Degrees of truth are taking from a finite chain $\mathcal{N}$.
- Axioms have a degree of truth associated.
- The fuzzy connectives used are a smooth t-norm $\otimes$ on $\mathcal{N}$, the strong negation $\ominus$ on $\mathcal{N}$, the dual t-conorm $\oplus$, and the implications $\Rightarrow_{s\otimes}, \Rightarrow_{r\otimes}, \Rightarrow_{ql\otimes}$.

In this paper, we will assume the reader to be familiar with classical DLs (for details, we refer to [1]).

3.1 Definition

Notation. In the rest of this paper, C, D are (possibly complex) concepts, A is an atomic concept, R is a role, a, b are individuals, $\bowtie \in \{\geq, <, \leq, >\}$, $\lhd \in \{\geq, >\}$, $\rhd \in \{\leq, <\}$. We will also use $\equiv$ to denote semantical equivalence, and we will not write $\otimes$ in the subscripts of the implications.

Syntax. Finite fuzzy $\mathcal{ALCH}$ assumes three alphabets of symbols, for concepts, roles and individuals. A *Fuzzy Knowledge Base* (KB) contains a finite set of axioms organized in a fuzzy ABox $\mathcal{A}$ (axioms about individuals), a fuzzy TBox $\mathcal{T}$ (axioms about concepts), and a fuzzy RBox $\mathcal{T}$ (axioms about roles).

The syntax of fuzzy concept, roles, and axioms are shown in Table 2. Note that in fuzzy $\mathcal{ALCH}$, all fuzzy roles are atomic.

Remark 1. As opposed to the crisp case, there are three types of universal restrictions, fuzzy GCIs, and fuzzy RIAs. In fact, the different subscripts $_s$, $_r$, and $_{ql}$ denote an S-implication, R-implication, and QL-implication, respectively.

Semantics. A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non empty set (the interpretation domain) and $\cdot^{\mathcal{I}}$ is a fuzzy interpretation function mapping *(i)* every individual a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, *(ii)* every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow \mathcal{N}$, and *(iii)* every role R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow \mathcal{N}$. The fuzzy interpretation function is extended to fuzzy *complex concepts* and *axioms* as shown in Table 2.

$C^{\mathcal{I}}$ denotes the membership function of the fuzzy concept C with respect to the fuzzy interpretation $\mathcal{I}$. $C^{\mathcal{I}}(x)$ gives us the degree of being x an element of the fuzzy concept C under $\mathcal{I}$. Similarly, $R^{\mathcal{I}}$ denotes the membership function of the fuzzy role R with respect to $\mathcal{I}$. $R^{\mathcal{I}}(x, y)$ gives us the degree of being (x, y) an element of the fuzzy role R.

Remark 2. Note an important difference with previous work in fuzzy DLs. Usually, $\cdot^{\mathcal{I}}$ maps every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$, and every role R onto $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$. Consequently, a fuzzy KB $\{\langle a : C > 0.5\rangle, \langle a : C < 0.75\}$ is satisfiable, by taking $C^{\mathcal{I}}(a) \in (0.5, 0.75)$. But now, given $\mathcal{N} = \{\texttt{false}, \texttt{closeToFalse}, \texttt{neutral}, \texttt{closeToTrue}, \texttt{true}\}$, a fuzzy KB $\{\langle a : C > \texttt{closeToFalse}\rangle, \langle a : C < \texttt{neutral}\}$ is not satisfiable, since $C^{\mathcal{I}}(a) \in \mathcal{N}$.

Witnessed models. In order to correctly manage infima and suprema in the reasoning, we need to define the notion of *witnessed* interpretations [7]. A fuzzy interpretation $\mathcal{I}$ is *witnessed* iff, for every formula, the infimum corresponds to the minimum and the supremum corresponds to the maximum. Our logic also enjoys the Witnessed Model Property (WMP) (all models are witnessed), because the number of degrees of truth in the models of the logic is finite [7].

Reasoning tasks. We will define the most important reasoning tasks and show that all of them can be reduced to fuzzy KB satisfiability.

- *Fuzzy KB satisfiability.* A fuzzy interpretation $\mathcal{I}$ *satisfies* (is a model of) a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R}\rangle$ iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.
- *Concept satisfiability.* C is α-satisfiable w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{\langle a : C \geq \alpha\rangle\}$ is satisfiable, where a is a new individual, which does not appear in $\mathcal{K}$.
- *Entailment.* A fuzzy concept assertion $\langle a : C \bowtie \alpha\rangle$ is entailed by a fuzzy KB $\mathcal{K}$ (denoted $\mathcal{K} \models \langle a : C \bowtie \alpha\rangle$) iff $\mathcal{K} \cup \{\langle a : C \neg\bowtie \alpha\rangle\}$ is unsatisfiable. Furthermore, $\mathcal{K} \models \langle (a, b) : R \geq \alpha\rangle$ iff $\mathcal{K} \cup \{\langle b : B \geq \gamma_p\rangle\} \models \langle a : \exists R.B \geq \alpha\rangle$, where B is a new concept.

Table 2. Syntax and semantics of finite fuzzy $\mathcal{ALCH}$

Element	Syntax	Semantics
Concepts	$\top$	γ_p
	$\bot$	γ_0
	A	$A^{\mathcal{I}}(x)$
	$C \sqcap D$	$C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$
	$C \sqcup D$	$C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$
	$\neg C$	$\ominus C^{\mathcal{I}}(x)$
	$\forall_s R.C$	$\inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow_s C^{\mathcal{I}}(y)\}$
	$\forall_r R.C$	$\inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow_r C^{\mathcal{I}}(y)\}$
	$\forall_{ql} R.C$	$\inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow_{ql} C^{\mathcal{I}}(y)\}$
	$\exists R.C$	$\sup_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\}$
Roles	R	$R^{\mathcal{I}}(x,y)$
ABox axioms	$\langle a : C \bowtie \gamma \rangle$	$C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \gamma$
	$\langle (a,b) : R \bowtie \gamma \rangle$	$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \bowtie \gamma$
TBox axioms	$\langle C \sqsubseteq_s D \rhd \gamma \rangle$	$\inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow_s D^{\mathcal{I}}(x)\} \rhd \gamma$
	$\langle C \sqsubseteq_r D \rhd \gamma \rangle$	$\inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow_r D^{\mathcal{I}}(x)\} \rhd \gamma$
	$\langle C \sqsubseteq_{ql} D \rhd \gamma \rangle$	$\inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow_{ql} D^{\mathcal{I}}(x)\} \rhd \gamma$
RBox axioms	$\langle R_1 \sqsubseteq_s R_2 \rhd \gamma \rangle$	$\inf_{x,y \in \Delta^{\mathcal{I}}}\{R_1^{\mathcal{I}}(x) \Rightarrow_s R_2^{\mathcal{I}}(x)\} \rhd \gamma$
	$\langle R_1 \sqsubseteq_r R_2 \rhd \gamma \rangle$	$\inf_{x,y \in \Delta^{\mathcal{I}}}\{R_1^{\mathcal{I}}(x) \Rightarrow_r R_2^{\mathcal{I}}(x)\} \rhd \gamma$
	$\langle R_1 \sqsubseteq_{ql} R_2 \rhd \gamma \rangle$	$\inf_{x,y \in \Delta^{\mathcal{I}}}\{R_1^{\mathcal{I}}(x) \Rightarrow_{ql} R_2^{\mathcal{I}}(x)\} \rhd \gamma$

- *Greatest lower bound*. The greatest lower bound of a concept or role assertion τ is defined as the $\sup\{\alpha : \mathcal{K} \models \langle \tau \geq \alpha \rangle\}$. It can be computed performing at most $\log |\mathcal{N}|$ entailment tests [16].
- *Concept subsumption*: Under an S-implication, D subsumes C with degree α ($C \sqsubseteq_s D \geq \alpha$) w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{a : \neg C \sqcup D < \alpha\}$ is unsatisfiable, where a is a new individual. Under an R-implication, D subsumes C ($C \sqsubseteq_r D$) w.r.t. a fuzzy KB $\mathcal{K}$ iff, for every $\alpha \in \mathcal{N}$, $\mathcal{K} \cup \{a : C \geq \alpha\} \cup \{a : D < \alpha\}$ is unsatisfiable, where a is a new individual. Under a QL-implication, D subsumes C with degree α ($C \sqsubseteq_{ql} D \geq \alpha$) w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{a : \neg C \sqcup (C \sqcap D) < \alpha\}$ is unsatisfiable, where a is a new individual.

3.2 Logical Properties

It can be easily shown that finite fuzzy $\mathcal{ALCH}$ is a sound extension of crisp $\mathcal{ALCH}$, because fuzzy interpretations coincide with crisp interpretations if we restrict the membership degrees to $\{\gamma_0 = 0, \gamma_p = 1\}$.

Proposition 7. *Finite fuzzy $\mathcal{ALCH}$ interpretations coincide with crisp interpretations if we restrict the membership degrees to $\{\gamma_0 = 0, \gamma_p = 1\}$.*

The following properties are extensions to a finite chain $\mathcal{N}$ of properties for Zadeh fuzzy DLs [3] and Łukasiewicz fuzzy DLs [6].

1. *Concept simplification*: $C \sqcap \top \equiv C$, $C \sqcup \bot \equiv C$,$C \sqcap \bot \equiv \bot$, $C \sqcup \top \equiv \top$, $\exists R.\bot \equiv \bot$, $\forall_s R.\top \equiv \top$, $\forall_r R.\top \equiv \top$, $\forall_{ql} R.\top \equiv \top$.
2. *Involutive* negation: $\neg\neg C \equiv C$,
3. *Excluded middle and contradiction*: In general, $C \sqcup \neg C \not\equiv \top$, $C \sqcap \neg C \not\equiv \bot$,
4. *Idempotence* of conjunction/disjunction: In general, $C \sqcap C \not\equiv C$, $C \sqcup C \not\equiv C$.

5. *De Morgan* laws: $\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$, $\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$,
6. *Inter-definability of concepts*: $\bot \equiv \neg\top$, $\top \equiv \neg\bot$, $C \sqcap D \equiv \neg(\neg C \sqcap \neg D)$, $C \sqcup D \equiv \neg(\neg C \sqcup \neg D)$, $\forall_s R.C \equiv \neg\exists R.(\neg C)$, $\exists R.C \equiv \neg\forall_s R.(\neg C)$. However, in general, $C \sqcap D \not\equiv \neg(\neg C \sqcup \neg D)$, $C \sqcup D \not\equiv \neg(\neg C \sqcap \neg D)$, $\forall_r R.C \not\equiv \neg\exists R.(\neg C)$, $\exists R.C \not\equiv \neg\forall_r R.(\neg C)$, $\forall_{ql} R.C \not\equiv \neg\exists R.(\neg C)$, $\exists R.C \not\equiv \neg\forall_{ql} R.(\neg C)$.
7. *Inter-definability of axioms*: $\langle \tau > \beta\rangle \equiv \langle \tau > +\beta\rangle$, $\langle \tau < \alpha\rangle \equiv \langle \tau \leq -\alpha\rangle$.
8. *Contrapositive symmetry*: $C \sqsubseteq_s D \equiv \neg D \sqsubseteq_s \neg C$. However, in general, $C \sqsubseteq_r D \not\equiv \neg D \sqsubseteq_r \neg_s C$,$C \sqsubseteq_{ql} D \not\equiv \neg D \sqsubseteq_{ql} \neg_s C$.
9. *Modus ponens*: $\langle a : C \triangleright \gamma_1\rangle$ and $\langle C \sqsubseteq_r D \triangleright \gamma_2\rangle$ imply $\langle a : D \triangleright \gamma_1 \otimes \gamma_2\rangle$, $\langle (a,b) : R \triangleright \gamma_1\rangle$ and $\langle R \sqsubseteq_r R' \triangleright \gamma_2\rangle$ imply $\langle (a,b) : R' \triangleright \gamma_1 \otimes \gamma_2\rangle$.
10. *Self-subsumption*: $(C \sqsubseteq_r C)^{\mathcal{I}} = \gamma_p$, $(R \sqsubseteq_r R)^{\mathcal{I}} = \gamma_p$. However, in general, $(C \sqsubseteq_s C)^{\mathcal{I}} \neq \gamma_p$, $(R \sqsubseteq_s R)^{\mathcal{I}} \neq \gamma_p$, and $(C \sqsubseteq_{ql} C)^{\mathcal{I}} \neq \gamma_p$, $(R \sqsubseteq_{ql} R)^{\mathcal{I}} \neq \gamma_p$.

Remark 3. Inter-definability of axioms makes it possible to restrict to fuzzy axioms of the forms $\langle \tau \geq \alpha\rangle$ and $\langle \tau \leq \beta\rangle$.

4 A Crisp Representation for Finite Fuzzy $\mathcal{ALCH}$

In this section we show how to reduce a fuzzy KB into a crisp KB. The procedure is satisfiability-preserving, so existing DL reasoners could be applied to the resulting KB. The basic idea is to create some new crisp concepts and roles, representing the α-cuts of the fuzzy concepts and relations, and to rely on them. Next, some new axioms are added to preserve their semantics and finally every axiom in the ABox, the TBox and the RBox is represented, independently from other axioms, using these new crisp elements.

Before proceeding formally, we will illustrate this idea with an example.

Example 3. Consider the smooth t-norm on $\mathcal{N}$ used in Example 1, and let us compute some α-cuts of the fuzzy concept $A_1 \sqcap A_2$ (denoted $\rho(A_1 \sqcap A_2, \geq \alpha)$).

To begin with, let us consider $\alpha = \gamma_2$. By definition, this set includes the elements of the domain x satisfying $A_1^{\mathcal{I}}(x) \otimes A_2^{\mathcal{I}}(x) \geq \gamma_2$. There are two possibilities: *(i)* $A_1^{\mathcal{I}}(x) \geq \gamma_2$ and $A_2^{\mathcal{I}}(x) \geq \gamma_3$, or *(ii)* $A_1^{\mathcal{I}}(x) \geq \gamma_3$ and $A_2^{\mathcal{I}}(x) \geq \gamma_2$. Hence, $\rho(A_1 \sqcap A_2, \geq \gamma_2) = \Big(\rho(A_1, \geq \gamma_2) \sqcap \rho(A_2, \geq \gamma_3)\Big) \sqcup \Big(\rho(A_1, \geq \gamma_3) \sqcap \rho(A_2, \geq \gamma_2)\Big)$.

Now, let us consider $\alpha = \gamma_3$. Now, there is only one possibility: $A_1^{\mathcal{I}}(a^{\mathcal{I}}) \geq \gamma_3$ and $A_2^{\mathcal{I}}(a^{\mathcal{I}}) \geq \gamma_3$. Hence, $\rho(A_1 \sqcap A_2, \geq \gamma_3) = \rho(A_1, \geq \gamma_3) \sqcap \rho(A_2, \geq \gamma_3)$.

Observe that for idempotent degrees ($\alpha \in J$) the case is the same as in finite Zadeh and Gödel fuzzy logics [3,5], whereas for non-idempotent degrees the case is similar as in finite Łukasiewicz fuzzy logic [6].

4.1 Adding New Elements

Let $\mathbf{A}$ be the set of atomic fuzzy concepts and $\mathbf{R}$ the set of atomic fuzzy roles in a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R}\rangle$, respectively. For each $\alpha \in \mathcal{N}^+$, for each $A \in \mathbf{A}$, a new atomic concepts $A_{\geq\alpha}$ is introduced. $A_{\geq\alpha}$ represents the crisp set of individuals which are instance of A with degree higher or equal than α i.e the α-cut of A. Similarly, for each $R \in \mathbf{R}$, a new atomic role $R_{\geq\alpha}$ is created.

Remark 4. The atomic elements $A_{\geq\gamma_0}$ and $R_{\geq\gamma_0}$ are not considered because they are always equivalent to the $\top$ concept. Also, as opposite to previous works [3,5,6] we are not introducing elements of the forms $A_{>\beta}$ and $R_{>\beta}$ (for each $\beta \in \mathcal{N} \setminus \{\gamma_p\}$), since now $A_{>\gamma_i}$ is equivalent to $A_{\geq\gamma_{i+1}}$, and $R_{>\gamma_i}$ is equivalent to $R_{\geq\gamma_{i+1}}$.

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leq i \leq p-1$ and for each $A \in \mathbf{A}$, $T(\mathcal{N})$ is the smallest terminology containing these axioms: $A_{\geq\gamma_{i+1}} \sqsubseteq A_{\geq\gamma_i}$. Similarly, for each $R_A \in \mathbf{R}$, $R(\mathcal{N})$ is the smallest terminology containing these axioms: $R_{\geq\gamma_{i+1}} \sqsubseteq R_{\geq\gamma_i}$.

Remark 5. Again, note that the number of new axioms needed here is less than the number needed in similar works [3,5,6], since we do not need to deal with elements of the forms $A_{>\beta}$ and $R_{>\beta}$.

4.2 Mapping Fuzzy Concepts, Roles and Axioms

Fuzzy concept and role expressions are reduced using mapping ρ, as shown in the top part of Table 3. Given a fuzzy concept C, $\rho(C, \geq \alpha)$ is a crisp set containing all the elements which belong to C with a degree greater or equal than α. The other cases $\rho(C, \bowtie \gamma)$ are similar. ρ is defined in a similar way for fuzzy roles. Furthermore, axioms are reduced as in the bottom part of Table 3, where $\kappa(\tau)$ maps a fuzzy axiom τ in finite fuzzy $\mathcal{ALCH}$ into a set of crisp axioms in $\mathcal{ALCH}$.

The reduction of the conjunction considers every pair $\gamma_x, \gamma_y \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$ such that $\alpha \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, and $x + y = i_{k+1} + z$, with $\alpha = \gamma_z$. Note that the reduction does not consider a closed internal of the form $[\gamma_{i_k}, \gamma_{i_{k+1}}]$. The reason is that, if α is idempotent and we set $\gamma_{i_{k+1}} = \alpha$, the result is correct ($\gamma_x = \gamma_y = \alpha$). However, setting $\gamma_{i_k} = \alpha$ would yield an incorrect result. Similarly, the reduction of the disjunction also considers a closed interval.

When dealing with R-implications and QL-implications, we consider optimal pairs of elements, to get efficient representation that avoids superfluous elements.

Definition 1. *Let $\Rightarrow$ be an implication in $\mathcal{N}$, and let $\gamma_x, \gamma_y \in \mathcal{N}^+$. (γ_x, γ_y) is a* $(\Rightarrow_{\geq\alpha})$-optimal *pair iff* (i) $\gamma_x \Rightarrow \gamma_y \geq \alpha$, (ii) *there are no $\gamma'_x, \gamma'_y \in \mathcal{N}^+$ such that $\gamma'_x \Rightarrow \gamma'_y \geq \alpha$, and such that either $\gamma'_x < \gamma_x$ or $\gamma'_y < \gamma_y$.*

Definition 2. *Let $\Rightarrow$ be an implication in $\mathcal{N}$, and let $\gamma_x \in \mathcal{N}^+, \gamma_y \in \mathcal{N}$. (γ_x, γ_y) is a* $(\Rightarrow_{\leq\beta})$-optimal *pair iff* (i) $\gamma_x \Rightarrow \gamma_y \leq \beta$, (ii) *there are no $\gamma'_x, \gamma'_y \in \mathcal{N}^+$ such that $\gamma'_x \Rightarrow \gamma'_y \leq \beta$, and such that either $\gamma'_x < \gamma_x$ or $\gamma'_y > \gamma_y$.*

Example 4. Given the R-implication in Example 2, the $(\Rightarrow_{\geq\gamma_3})$-optimal pairs are (γ_3, γ_3), (γ_2, γ_2), and (γ_1, γ_1); and the $(\Rightarrow_{\leq\gamma_3})$-optimal pairs are (γ_5, γ_3), (γ_3, γ_2), (γ_2, γ_1), and (γ_1, γ_0).

Note that R-implications are, in general, non smooth (see Example 2). Hence, a pair of elements γ_1, γ_y such that $\gamma_x \Rightarrow_r \gamma_y = \alpha$ might not exist, and thus we have to consider an inequality of the form $\gamma_x \Rightarrow_r \gamma_y \geq \alpha$. In QL-implications, due to the optimality condition, $=$ and $\geq$ yield the same result.

Table 3. Mapping of concepts, roles, and axioms

$\rho(\top, \geq \alpha)$ $\rho(\top, \leq \beta)$	$\top$ $\bot$
$\rho(\bot, \geq \alpha)$ $\rho(\bot, \leq \beta)$	$\bot$ $\top$
$\rho(A, \geq \alpha)$ $\rho(A, \leq \beta)$	$A_{\geq \alpha}$ $\neg A_{\geq +\beta}$
$\rho(\neg C, \bowtie \gamma)$	$\rho(C, \bowtie^{-} \ominus \gamma)$
$\rho(C \sqcap D, \geq \alpha)$	$\bigsqcup_{\gamma_x,\gamma_y}\{\rho(C, \geq \gamma_x) \sqcap \rho(D, \geq \gamma_y)\}$ for every pair γ_x, γ_y such that $\alpha, \gamma_x, \gamma_y \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, and $x + y = i_{k+1} + z$, with $\gamma_z = \alpha$
$\rho(C \sqcap D, \leq \beta)$	$\rho(\neg C \sqcup \neg D, \geq \ominus \beta)$
$\rho(C \sqcup D, \geq \alpha)$	$\rho(C, \geq \alpha) \sqcup \rho(D, \geq \alpha) \bigsqcup_{\gamma_x,\gamma_y}\{\rho(C, \geq \gamma_x) \sqcap \rho(D, \geq \gamma_y)\}$ for every pair γ_x, γ_y such that $\alpha, \gamma_x, \gamma_y \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, and $x + y = i_k + z$, with $\gamma_z = \alpha$
$\rho(C \sqcup D, \leq \beta)$	$\rho(\neg C \sqcap \neg D, \geq \ominus \beta)$
$\rho(\exists R.C, \geq \alpha)$	$\bigsqcup_{\gamma_x,\gamma_y}\{\exists \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$ for every pair $\gamma_x, \gamma_y \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$ such that $\gamma \in (\gamma_{i_k}, \gamma_{i_{k+1}}]$, and $x + y = i_{k+1} + z$, with $\gamma_z = \alpha$
$\rho(\exists R.C, \leq \beta)$	$\rho(\forall_s R.(\neg C), \geq \ominus \beta)$
$\rho(\forall_s R.C, \geq \alpha)$	$\bigsqcap_{\gamma_x,\gamma_y}\{\forall \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$ for every pair γ_x, γ_y such that $\gamma_x \in (\gamma_{i_k}, \gamma_{i_{k+1}}], \alpha, \gamma_y \in (\gamma_{p-i_{k+1}}, \gamma_{p-i_k}]$, and $y - i = z - i_{k+1}$, with $\gamma_z = \alpha$
$\rho(\forall_s R.C, \leq \beta)$	$\rho(\exists R.(\neg C), \geq \ominus \beta)$
$\rho(\forall_r R.C, \geq \alpha)$	$\bigsqcap_{\gamma_x,\gamma_y}\{\forall \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$ for every pair $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{r\ \geq \alpha})$-optimal
$\rho(\forall_r R.C, \leq \beta)$	$\bigsqcup_{\gamma_x,\gamma_y}\{\exists \rho(R, \geq \gamma_x).\rho(C, \leq \gamma_y)\}$ for every pair $\gamma_x \in \mathcal{N}^+, \gamma_y \in \mathcal{N}$ such that γ_x, γ_y are $(\Rightarrow_{r\ \leq \beta})$-optimal
$\rho(\forall_{ql} R.C, \geq \alpha)$	$\bigsqcap_{\gamma_x,\gamma_y}\{\forall \rho(R, \geq \gamma_x).\rho(C, \geq \gamma_y)\}$ for every pair $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{ql\ \geq \alpha})$-optimal
$\rho(\forall_{ql} R.C, \leq \beta)$	$\bigsqcup_{\gamma_x,\gamma_y}\{\exists \rho(R, \geq \gamma_x).\rho(C, \leq \gamma_y)\}$ for every pair $\gamma_x \in \mathcal{N}^+, \gamma_y \in \mathcal{N}$ such that γ_x, γ_y are $(\Rightarrow_{ql\ \leq \beta})$-optimal
$\rho(R, \geq \alpha)$ $\rho(R, \leq \beta)$	$R_{\geq \alpha}$ $\neg R_{\geq +\beta}$
$\kappa(\langle a : C \bowtie \gamma \rangle)$	$\{a : \rho(C, \bowtie \gamma)\}$
$\kappa(\langle (a,b) : R \bowtie \gamma \rangle)$	$\{(a,b) : \rho(R, \bowtie \gamma)\}$
$\kappa(\langle C \sqsubseteq_s D \geq \alpha \rangle)$	$\bigcup\{\rho(C, \geq \gamma_x) \sqsubseteq \rho(D, \geq \gamma_y)\}$ for every pair γ_x, γ_y such that $\gamma_x \in (\gamma_{i_k}, \gamma_{i_{k+1}}], \alpha, \gamma_y \in (\gamma_{p-i_{k+1}}, \gamma_{p-i_k}]$, and $y - \gamma_i = z - \gamma_{i_{k+1}}$, with $\gamma_z = \alpha$
$\kappa(\langle C \sqsubseteq_r D \geq \alpha \rangle)$	$\bigcup\{\rho(C, \geq \gamma_x) \sqsubseteq \rho(D, \geq \gamma_y)\}$ for every pair $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{r\ \geq \alpha})$-optimal
$\kappa(\langle C \sqsubseteq_{ql} D \geq \alpha \rangle)$	$\bigcup\{\forall \rho(C, \geq \gamma_x) \sqsubseteq \rho(D, \geq \gamma_y)\}$ for every pair $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{ql\ \geq \alpha})$-optimal
$\kappa(\langle R_1 \sqsubseteq_s R_2 \geq \alpha \rangle)$	$\bigcup\{\rho(R_1, \geq \gamma_x) \sqsubseteq \rho(R_2, \geq \gamma_y)\}$ for every pair γ_x, γ_y such that $\gamma_x \in (\gamma_{i_k}, \gamma_{i_{k+1}}], \alpha, \gamma_y \in (\gamma_{p-i_{k+1}}, \gamma_{p-i_k}]$, and $y - \gamma_i = z - \gamma_{i_{k+1}}$, with $\gamma_z = \alpha$
$\kappa(\langle R_1 \sqsubseteq_r R_2 \geq \alpha \rangle)$	$\bigcup\{\rho(R_1, \geq \gamma_x) \sqsubseteq \rho(R_2, \geq \gamma_y)\}$ for every pair $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{r\ \geq \alpha})$-optimal
$\kappa(\langle R_1 \sqsubseteq_{ql} R_2 \geq \alpha \rangle)$	$\bigcup\{\rho(R_1, \geq \gamma_x) \sqsubseteq \rho(R_2, \geq \gamma_y)\}$ for every pair $\gamma_x, \gamma_y \in \mathcal{N}^+$ such that γ_x, γ_y are $(\Rightarrow_{ql\ \geq \alpha})$-optimal

$\kappa(\mathcal{A})$ (resp. $\kappa(\mathcal{T})$, $\kappa(\mathcal{R})$) denotes the union of the reductions of every axiom in $\mathcal{A}$ (resp. $\mathcal{T}$, $\mathcal{R}$). $\mathsf{crisp}(\mathcal{K})$ denotes the reduction of a fuzzy KB $\mathcal{K}$. A fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ is reduced into a KB $\mathsf{crisp}(\mathcal{K}) = \langle \kappa(\mathcal{A}), T(\mathcal{N}) \cup \kappa(\mathcal{T}), R(\mathcal{N}) \cup \kappa(\mathcal{R}) \rangle$.

4.3 Properties of the Reduction

Correctness. The following theorem, showing the logic is decidable and that the reductions preserves reasoning, can be shown.

Theorem 1. *The satisfiability problem in finite fuzzy $\mathcal{ALCH}$ is decidable. Furthermore, a finite fuzzy $\mathcal{ALCH}$ fuzzy KB $\mathcal{K}$ is satisfiable iff $\mathsf{crisp}(\mathcal{K})$ is.*

Complexity. In general, the size of $\mathsf{crisp}(\mathcal{K})$ is $\mathcal{O}(|\mathcal{K}| \cdot |\mathcal{N}|^k)$, being k the maximal depth of the concepts appearing in $\mathcal{K}$. In the particular case of finite Zadeh fuzzy logic, the size of $\mathsf{crisp}(\mathcal{K})$ is $\mathcal{O}(|\mathcal{K}| \cdot |\mathcal{N}|)$ [3]. For other fuzzy operators the case is more complex because we cannot infer the exact values of the degrees of truth, so we need to build disjunctions or conjunctions over all possible degrees of truth.

Modularity. The reduction of an ontology can be reused when adding new axioms if they do not introduce new atomic concepts and roles. In this case, it remains to add the reduction of the new axioms. This allows to compute the reduction of the ontology off-line and update $\mathsf{crisp}(\mathcal{K})$ incrementally. The assumption that the basic vocabulary is fully expressed in the ontology is reasonable because ontologies do not usually change once that their development has finished.

5 Conclusions and Future Work

This paper has set a general framework for fuzzy DLs with a finite chain of degrees of truth $\mathcal{N}$. $\mathcal{N}$ can be seen as a finite totally ordered set of linguistic terms or labels. This is very useful in practice, since expert knowledge is usually expressed using linguistic terms and avoiding their numerical interpretations.

Starting from a smooth finite t-norm on $\mathcal{N}$, we define the syntax and semantics of fuzzy $\mathcal{ALCH}$. The negation function and the t-conorm are imposed by the choice of the t-norm, but there are different options for the implication function. For this reason, whenever this is possible (i.e., in universal restriction concepts and in inclusion axioms), the language allows to use three different implications. We have studies some of the logical properties of the logic. This will help the ontology developers to use the implication that better suit their needs.

The decidability of the logic has been shown by presenting a reasoning preserving reduction to the crisp case. Providing a crisp representation for a fuzzy ontology allows reusing current crisp ontology languages and reasoners, among other related resources. The complexity of the crisp representation is higher than in finite Zadeh fuzzy DLs, because it is necessary to build disjunctions or conjunctions over all possible degrees of truth. However, Zadeh fuzzy DLs have some logical problems [3] which may not be acceptable in some applications, where alternative operators such as those introduced in this paper could be used.

As future work we will study more expressive logics than $\mathcal{ALCH}$, applying the ideas in the previous work DLs [3,5,6], with the aim of providing the theoretical basis of a fuzzy extension of OWL 2 under finite chain of degrees of truth.

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