Tractable Reasoning Based on the Fuzzy $\mathcal{EL}^{++}$ Algorithm

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Uncertainty Reasoning for the Semantic Web

Outline

Fuzzy DLs

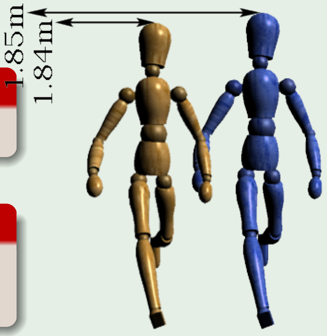
Fuzzy DLs = **Classic DLs** + **Uncertainty and Imprecision**

Crisp environment:

$\neg Tall(Mairy), Tall(John)$

Fuzzy environment:

$Tall(Mairy) \geq 0.75$,
$Tall(John) \geq 0.8$



Motivation

- Most Fuzzy DL algorithms are intractable
  $\Longrightarrow$Inapropriate for scalable problems
- We propose a fuzzy extension of the EL++ algorithm

Related Work

- Peter Vojtás. EL description logic with aggregation of user preference concepts.
- Giorgos Stoilos, Giorgos B. Stamou, Jeff Z. Pan: Classifying Fuzzy Subsumption in Fuzzy-EL+.
- Umberto Straccia. Answering Vague Queries in Fuzzy DL-LITE.
- Jeff Z. Pan, Giorgos B. Stamou, Giorgos Stoilos, Edward Thomas: Expressive Querying over Fuzzy DL-Lite Ontologies.

Structural Elements of $\mathcal{EL}^{++}$

- Individual Names
  - E.g. *John*, *Mairy*, *Table*123, *Chair*232, ...
- Concepts Names
  - E.g. *Tall*, *Short*, *Happy*, ...
- Role Names
  - E.g. *hasFriend*, *likes*, *isInLoveWith*, ...
- Concept Constructors $(\top, \bot, \{a\}, C \sqcap D, \exists r.C)$
  - E.g. $Tall \sqcap Beautiful$, $\exists hasFriend.Beautiful$, $\{John\}$

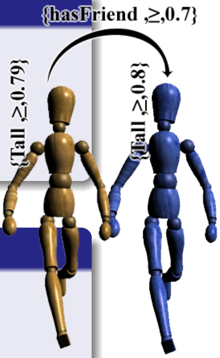
$\mathcal{EL}^{++}$ Knowledge Base

Assertional Box (ABox) $\mathcal{A}$

Concept Assertions $C(a) \geq d$
$Tall(John) \geq 0.8$
Role Assertions $r(a, b) \geq d$
$hasFriend(John, Mairy) \geq 0.7$

Constraint Box (CBox) $\mathcal{C}$

General fuzzy concept inclusions $C \sqsubseteq^d D$
$Tall \sqcap GoodDribbler \sqsubseteq^{0.7} PlaysGoodBasketBall$
Role inclusion axioms $r_1 \circ \ldots \circ r_n \sqsubseteq s$
$hasFriend \circ hasFriend \sqsubseteq hasFriend$



Semantics

The semantics of an $\mathcal{EL}^{++}$ are given via a fuzzy interpretation $\mathcal{I} = \{\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\}$ where the domain $\Delta^{\mathcal{I}}$ is a set of objects and the interpretation function $\cdot^{\mathcal{I}}$ maps:

- each individual a to an element in $\Delta^{\mathcal{I}}$
- each concept name A to a membership function $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$
- each role name r to a membership function $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$

Semantics

Syntax	Semantics
$\top$	$\top^{\mathcal{I}}(x) = 1$
$\bot$	$\bot^{\mathcal{I}}(x) = 0$
$\{a\}$	$\{a\}^{\mathcal{I}}(x) = \begin{cases} 1 & \text{when } x = a^{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}$
$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}}(x) = min\left(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\right)$
$\exists r.C$	$(\exists r.C)^{\mathcal{I}}(x) = sup_{y \in \Delta^{\mathcal{I}}}\left(min\left(r^{\mathcal{I}}(x,y), C^{\mathcal{I}}(y)\right)\right)$
$C \sqsubseteq^{d} D$	$min\left(C^{\mathcal{I}}(x), d\right) \leq D^{\mathcal{I}}(x)$
$r_1 \circ \ldots \circ r_k \sqsubseteq s$	$\left[r_1^{\mathcal{I}} \circ^{t} \ldots \circ^{t} r_k^{\mathcal{I}}\right](x,y) \leq s^{\mathcal{I}}(x,y)$
$C(a) \geq d$	$C^{\mathcal{I}}\left(a^{\mathcal{I}}\right) \geq d$
$r(a,b) \geq d$	$r^{\mathcal{I}}\left(a^{\mathcal{I}}, b^{\mathcal{I}}\right) \geq d$

Reasoning in fuzzy $\mathcal{EL}^{++}$

Reduce each inference problem to a fuzzy subsumption problem

$\Downarrow$

Transform an $\mathcal{EL}^{++}$ CBox to its Normal Form

$\Downarrow$

Apply the fuzzy Subsumption Algorithm for $\mathcal{EL}^{++}$

Step 1: Reducing each Inference Problem to Concept Subsumption

- *Concept satisfiability*
  - $C \not\sqsubseteq^1_{\mathcal{C}} \bot$.
- *Instance Problem*
  - We convert an ABox $\mathcal{A}$ to a set of fuzzy GCIs $\mathcal{C}_{\mathcal{A}}$: $\mathcal{C}_{\mathcal{A}} = \{\{a\} \sqsubseteq^d C \mid (C(a) \geq d) \in \mathcal{A}\} \bigcup \{\{a\} \sqsubseteq^d \exists r.\{b\} \mid (r(a,b) \geq d) \in \mathcal{A}\}$.
  - Then a is an instance of a concept C with a degree at least d iff $\{a\} \sqsubseteq^d_{\mathcal{C} \cup \mathcal{C}_{\mathcal{A}}} C$.
- *ABox consistency*
  - We convert an ABox $\mathcal{A}$ to a set of fuzzy GCIs $\mathcal{C}_{\mathcal{A}}$ as before
  - We create a concept $C_{\mathcal{A}}$ as follows $C_{\mathcal{A}} = \sqcap_{a \in N_I} \exists u.\{a\}$,
  - Then $\mathcal{A}$ iff $C_{\mathcal{A}} \not\sqsubseteq_{\mathcal{C} \cup \mathcal{C}_{\mathcal{A}}} \bot$.

Step 2: Normalization of fuzzy $\mathcal{EL}^{++}$ CBoxes

Normal Form of CBoxes

$$
\begin{array}{rcl}
C_1 & \sqsubseteq^d & D \\
C_1 \sqcap C_2 & \sqsubseteq^d & D \\
C_1 & \sqsubseteq^d & \exists r.C_2 \\
\exists r.C_1 & \sqsubseteq^d & D \\
r_1 & \sqsubseteq^d & s \\
r_1 \circ r_2 & \sqsubseteq^d & s
\end{array}
$$

Step 2: Normalization of fuzzy $\mathcal{EL}^{++}$ CBoxes

$$Tall \sqcap \exists hasFather.BasketBallCoach \sqsubseteq^{0.8} PlaysGoodBasketBall$$
$$\Downarrow$$
$$\overbrace{\begin{array}{c}\exists hasFather.BasketBallCoach \sqsubseteq^{0.8} A \\ Tall \sqcap A \sqsubseteq^{0.8} PlaysGoodBasketBall\end{array}}$$

$$\{John\} \sqsubseteq^{0.8} Tall \sqcap GoodDribbler$$
$$\Downarrow$$
$$\overbrace{\begin{array}{c}\{John\} \sqsubseteq^{0.8} GoodDribbler \\ \{John\} \sqsubseteq^{0.8} Tall\end{array}}$$

Step 3: Fuzzy Subsumption Algorithm for $\mathcal{EL}^{++}$

Our algorithm is based on two mappings:

- a mapping S from $BC_{\mathcal{C}} \times BC_{\mathcal{C}}$ to $[0, 1]$
- a mapping R from $R_{\mathcal{C}} \times BC_{\mathcal{C}} \times BC_{\mathcal{C}}$ to $[0, 1]$

Each of these two mappings has the purpose of making implicit fuzzy subsumption relationships, explicit:

- $S(C, D) = d$ implies that $C \sqsubseteq_{\mathcal{C}}^{d} D$
- $R(r, C, D) = d$ implies that $C \sqsubseteq_{\mathcal{C}}^{d} \exists r.D$

Step 3: Fuzzy Subsumption Algorithm for $\mathcal{EL}^{++}$

$$S(\{John\}, Strong) = 0.7\ ,\ S(\{John\}, Tall) = 0.6\ ,$$
$$Strong \sqcap Tall \sqsubseteq^{0.6} PlaysGoodBasketBall$$
$$\Downarrow$$
$$S(\{John\}, PlaysGoodBasketBall) = 0.6$$

$$R(hasFriend, John, Mairy) = 0.7,$$
$$R(hasFriend, Mairy, Anna) = 0.5,$$
$$hasFriend \circ hasFriend \sqsubseteq hasFriend$$
$$\Downarrow$$
$$R(hasFriend, John, Mairy) = 0.5$$

Contribution and Future Work

Contribution

- Introduction of Nominals and the bottom concept that allow
  - to perform reasoning with assertional knowledge
  - to answer to the concept satisfiability and ABox consistency problems
  - to express disjointness between concepts
- We describe how each problem in fuzzy EL++ can be reduced to the fuzzy Concept Subsumption Problem

Future Work

- Study of concrete domains
- Study of domain and range properties

Normalization of fuzzy $\mathcal{EL}^{++}$ CBoxes

Normalization Rules

NF1	$r_1 \circ \ldots \circ r_k \sqsubseteq s$	$\longrightarrow$	$r_1 \circ \ldots \circ r_{k-1} \sqsubseteq u,\ u \circ r_k \sqsubseteq s$
NF2	$C \sqcap \hat{D} \sqsubseteq^d E$	$\longrightarrow$	$\hat{D} \sqsubseteq^d A,\ C \sqcap A \sqsubseteq^d E$
NF3	$\exists r.\hat{C} \sqsubseteq^d D$	$\longrightarrow$	$\hat{C} \sqsubseteq^d A,\ \exists r.A \sqsubseteq^d D$
NF4	$\bot \sqsubseteq^d D$	$\longrightarrow$	$\emptyset$
NF5	$\hat{C} \sqsubseteq^d \hat{D}$	$\longrightarrow$	$\hat{C} \sqsubseteq^d A,\ A \sqsubseteq^d \hat{D}$
NF6	$B \sqsubseteq^d \exists r.\hat{C}$	$\longrightarrow$	$B \sqsubseteq^d \exists r.A,\ A \sqsubseteq^d \hat{C}$
NF7	$B \sqsubseteq^d C \sqcap D$	$\longrightarrow$	$B \sqsubseteq^d C,\ B \sqsubseteq^d D$

Fuzzy Subsumption Algorithm for $\mathcal{EL}^{++}$

Completion Rules

If $S(C, C') = d_1,\ C' \sqsubseteq^{d_2} D \in \mathcal{C}$ and $S(C, D) < min(d_1, d_2)$
then $S(C, D) = min(d_1, d_2)$
If $S(C, C_1) = d_1,\ S(C, C_2) = d_2\ ,\ C_1 \sqcap C_2 \sqsubseteq^{d_3} D \in \mathcal{C},$
and $S(C, D) < min(d_1, d_2, d_3)$
then $S(C) := min(d_1, d_2, d_3)$
If $S(C, C') = d_1,\ C' \sqsubseteq^{d_2} \exists r.D \in \mathcal{C}$ and $R(r, C, D) < min(d_1, d_2)$
then $R(r, C, D) := min(d_1, d_2)$
If $R(r, C, D) = d_1,\ S(D, C') = d_2,\ \exists r.C' \sqsubseteq^{d_3} E \in \mathcal{C}$
and $S(C, E) < min(d_1, d_2, d_3)$
then $S(C, E) = min(d_1, d_2, d_3)$
If $R(r, C, D) > 0,\ S(D, \bot) > 0$ and $S(C, \bot) = 0,$
then $S(C, \bot) = 1$

Fuzzy Subsumption Algorithm for $\mathcal{EL}^{++}$

Completion Rules

If $S(C, \{a\}) = 1$, $S(E, \{a\}) = 1$ and $C \leadsto_d E$,
then for each $D \in BC_{\mathcal{C}}$, if $S(C, D) < min(d, S(E, D))$
$S(C, D) := min(d, S(E, D))$
If $R(r, C, D) = d$, $r \sqsubseteq s \in \mathcal{C}$ and $R(s, C, D) < d$
then $R(s, C, D) := d$
If $R(r_1, C, D) = d_1$, $R(r_2, D, E) = d_2$, $r_1 \circ r_2 \sqsubseteq r_3 \in \mathcal{C}$
and $R(r_3, C, E) < min(d_1, d_2)$
then $\mathrm{R}(r_3, C, D) := min(d_1, d_2)$
If $S(C, \{a\}) > 0$ for some nominal $\{a\}$ and $S(C, \{a\}) < 1$
then $S(C, \{a\}) := 1$