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# An Application of Interacting Multiple Model Tracking Method to Financial Modeling and Asset Allocation

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***Abstract - This paper describes a continuous-time-state-process, discrete-time-observation, Interacting Multiple Model (IMM) tracking algorithm, and its applications to financial market modeling and asset allocation. A system state is modeled as a continuous-time, affine-Gaussian stochastic dynamical process driven by a white process noise, as well as structural changes modeled by a finite-state, continuous-time, Markov process. The system generally assumes multiple models with different state space dimensions and an affine-Gaussian state jump whenever a model transition occurs. The underlying problem is a standard filtering problem for estimating the system state based on a sequence of discrete-time, linear-Gaussian observations of partial system states. As our first attempt for applying the IMM methods to financial market modeling, we will use a rather naïve switching process using simple multiple linear stochastic system models.***

**Keywords:** Interacting Multiple Model (IMM) Tracking, Dynamic State Estimation, Markov Jump Linear Systems, Switching Financial Market Modeling, Asset Allocation.

## 1 Introduction

In this paper, we are generally concerned with financial market modeling and asset allocation problems, and specifically with the possibilities of Interacting Multiple Model (IMM) methods (which were developed as maneuver models for target tracking [1] in 1980s, and since then, have been refined in many directions) being applied to the financial market modeling. This paper will expand the continuous-time IMM extrapolation algorithm of [14] with a typical tracking stop-and-go target example, to a full IMM algorithm. This algorithm will then be used to analyze financial market behaviors that we model as a continuous-time stochastic dynamical system with discrete-time observations, in which the system structures are switched among multiple models.

Since the time when the IMM approach to tracking maneuvering targets was first published ([4,5]), the IMM methods have been widely used to make tracking algorithms adaptive to a wide range of maneuvering and other abrupt structural changes in target motion dynamics. In fact, the IMM algorithms are one of the most well-studied subjects in target tracking, as documented in [3-7]. As a target tracking algorithm, each model used in an IMM algorithm typically represents a standard target behavior such as an almost(nearly in [1])-constant-velocity model, and an almost-constant-rate turn model. Alternatively, multiple models may represent different levels of white process noises in the target dynamics so as to expand the range of tracking (filtering) bandwidth adaptively.

In a typical IMM implementation, both model switching and state transition are allowed to happen only on prescribed discrete time steps. Indeed, almost all the IMM literature starts with a discrete-time target dynamics formalism. A few exceptions include [7,8], in which the target dynamics are described by stochastic differential equations driven by Poisson processes as well as Wiener processes. This paper uses the mathematical model, described in [14], expressed by a continuous-time Markov process on a hybrid state space explicitly through the semi-group of state transition operators and its infinitesimal generator. Like the model described in [9], our model allows switching across spaces with different dimensions, and as in [7,10], our model allows the system state to jump whenever a model switching happens. These flexibilities have motivated us to explore the possibilities of applications to financial market modeling. As expressed in [17], our general motivation is to explore possibilities of applications of engineering techniques to the social and economic system analysis.

In the next section, Section 2, we will define a continuous-time Markov jump linear/affine system as a Markovian process on a hybrid state space with a continuous time parameter, and based on it, we will define a filtering problem, a solution to which is given in Section 3, where an IMM algorithm, with the continuous time extrapolation and discrete time updating, will be described. Section 4 shows a simple three-model financial market model with an IMM extrapolation algorithm. Numerical examples of financial market modeling and asset allocation analysis will be shown in Section 5, followed by our conclusions in Section 6.

## 2 Jump Markov Model

Consider $M$ models, each of which is defined by a vector-matrix triple $(A_m, b_m, B_m)$ that defines an Itô's linear or affine stochastic differential equation as $dx_t = (A_m x_t + b_m)dt + B_m dw_t$, $m = 1,\dots,M$, which defines a continuous-time stochastic process $x_t$ on a Euclidean space $E_m$, with a vector-valued, unit-intensity Wiener process $w_t$, on an appropriate time interval. Thus, within a model $m$, the target state $x_t$ is continuous (no jump).

We assume that model transition is expressed by a continuous-time, $\{1,\dots,M\}$-valued, time-homogeneous Markov process $(m_t)_{t\in[t_0,\infty)}$ with transition probability

$$P_h(m' \mid m) \overset{\text{def}}{=} \text{Prob}\{m_{t+h} = m' \mid m_t = m\} = \begin{cases} c_{mm'}h + o(h) & \text{if } m' \neq m \\ 1 - \sum_{\substack{m'=1 \\ m'\neq m}}^{M} c_{mm'}h + o(h) & \text{otherwise} \end{cases} \quad (1)$$

for $m, m' = 1,\dots,M$, $h > 0$, and $t \in [t_0,\infty)$, with constants $c_{mm'} \geq 0$ for $m' \neq m$, $-c_{mm} = \sum_{\substack{m'=1 \\ m'\neq m}}^{M} c_{mm'} \quad 0$, and a fixed $t_0$. We assume each model transition is accompanied with an affine-Gaussian jump. Namely, when a model transition from $m$ to $m'$ happens at time $t$, the target state jumps from $\lim_{h\downarrow 0} x_{t-h}$ in $E_m$ to[1] $x_t = \lim_{h\downarrow 0} x_{t+h}$ that is a generalized Gaussian vector with mean vector $F_m^{m'} \lim_{h\downarrow 0} x_{t-h} + g_m^{m'}$ and a positive semi-definite covariance matrix $V_m^{m'}$ in $E_{m'}$, where $F_m^{m'}$ and $g_m^{m'}$ are a matrix and a vector with appropriate dimensions. We use the convention that $F_m^m = I$ (the identity matrix), $g_m^m = 0$ (the zero vector), and $V_m^m = 0$ (the zero matrix) for each $m$, preventing any jump within the same model.

A more precise mathematical model can be expressed as a continuous-time, time-homogeneous Markov process $(x_t, m_t)_{t\in[t_0,\infty)}$ on a hybrid state space[2] $E \overset{\text{def}}{=} \bigcup_{m=1}^{M} E_m \times \{m\}$ that is a formal direct-sum of Euclidean spaces $E_m$ with generally different dimensions, with a transition probability

$$\begin{aligned} &\text{Prob}\{x_{t+h_1+h_2} \in dx', m_{t+h_1+h_2} = m' \mid x_t \quad x, m_t \quad m\} \quad P_{h_1+h_2}(m' \mid m) \\ &\int_{E_m}\int_{E_{m'}} \mathcal{G}(dx'; \Delta F_{m'}(h_2)x'' + \Delta g_{m'}(h_2), \Delta V_{m'}(h_2)) \\ &\qquad \mathcal{G}(dx''; F_m^{m'}x''' + g_m^{m'}, V_m^{m'}) \\ &\qquad \mathcal{G}(dx'''; \Delta F_m(h_1)x + \Delta g_m(h_1), \Delta V_m(h_1)) + o(h_1+h_2) \end{aligned} \quad (2)$$

for each $m, m' = 1,\dots,M$, each $x \in E_m$, each $t \in [t_0,\infty)$, and $h_1, h_2 > 0$, where[3], for each $m$ and $h \geq 0$, $\Delta F_m(h) \overset{\text{def}}{=} e^{A_m h}$, $\Delta g_m(h) \overset{\text{def}}{=} \int_0^h e^{A_m \tau} b_m d\tau$, and $\Delta V_m(h) \overset{\text{def}}{=} \int_0^h e^{A_m \tau} Q_m e^{A_m^T \tau} d\tau$ with $Q_m = B_m B_m^T$. $\mathcal{G}(\cdot; \bar{\xi}, V)$ is the symbol for the generic generalized Gaussian distribution with mean vector $\bar{\xi}$ and a positive semi-definite covariance matrix $V$, of compatible dimensions, defined by its characteristic function as

$$\int_{E_m} e^{\sqrt{-1}\zeta^T \xi} \mathcal{G}(d\xi; \bar{\xi}, V) = \exp\left(\sqrt{-1}\bar{\xi}^T \zeta - \frac{1}{2}\zeta^T V \zeta\right) \quad (3)$$

for each vector $\zeta$ with the dimension determined by the parameter pair $(\bar{\xi}, V)$.

The discrete time observations, $y_1, y_2, y_3, \dots$, are defined by

$$y_k = H_{m_{t_k}k} x_{t_k} + \eta_k \quad (4)$$

$k = 1,2,3,\dots$, for the time sequence, $t_1, t_2, t_3, \dots$, such that $t_0 \leq t_k < t_{k+1}$ for each $k$, with observation matrices, $(H_{mk})_{m=1}^{M}$, $k = 1,2,3,\dots$, of appropriate dimensions, and with zero-mean independent Gaussian vectors $\eta_1, \eta_2, \eta_3, \dots$, with covariance matrices[4] $R_k = \mathbb{E}(\eta_k \eta_k^T)$. The independent initial condition is given as

$$\text{Prob}\left\{x_{t_0} \in dx, m_{t_0} = m\right\} = p_{m0}\mathcal{G}(dx; \bar{x}_{m0}, \bar{V}_{m0}) \quad (5)$$

with an initial model probability $p_{m0}$, mean $\bar{x}_{m0}$, a positive definite covariance matrix $\bar{V}_{m0}$, for each $m$, and a time index $t_0$ such that $t_0 \leq t_1$.

Then the filtering problem defined by eqns. (1) to (5) is the problem for characterizing the a posteriori probability distribution, expressed by $\hat{p}_{mk}$=$\text{Prob}\{m_{t_k} = m \mid y_1,\dots,y_k\}$ and $\text{Prob}\{x_{t_k} \in dx_{t_k} \mid m_{t_k} = m, y_1,\dots,y_k\}$ for each $m = 1,\dots,M$, each $k = 1,2,3,\dots$ It would be extremely difficult (if not impossible) to express $\text{Prob}\{x_{t_k} \in dx_{t_k} \mid m_{t_k} = m, y_1,\dots,y_k\}$ in any analytical (closed) form because of the infinity possibilities of how the system jumps occur, in any given interval $\left[t_{k-1} \quad t_k\right]$.

Like in [14], therefore, we will only consider the posterior model probability $\hat{p}_{mk}$, and the first and the second moments of the posterior state probability

[1] We assume the right-continuity, to eliminate any ambiguity.

[2] Since $E = \Re^n \times \{1,\dots,M\}$ if $E_m = \Re^n$ for all $m = 1,\dots,M$, our choice of the state space provides a proper extension to the usual models used for multiple-model formulations.

[3] By $X^T$ we mean the transpose of a vector or a matrix $X$.

[4] $\mathbb{E}$ is the symbol for the conditional and the unconditional mathematical expectation operators.

distribution, $\text{Prob}\{x_{t_k} \in dx_{t_k} \mid m_{t_k} = m, y_1, ..., y_k\}$, given model $m$, i.e., $\hat{x}_{mk} = \mathbb{E}(x_{t_k} \mid m_{t_k} = m, y_1, ..., y_k)$ and $\hat{V}_{mk} = \mathbb{E}(x_{t_k} x_{t_k}^T \mid m_{t_k} = m, y_1, ..., y_k) - \hat{x}_{mk}\hat{x}_{mk}^T$.

*Remark 1*: The IMM filtering problem described in [7] is actually a continuous-time-dynamic-system, continuous-time-observation problem, where inter-model jumps are modeled by an affine function with coefficients that are functions of a Poisson random measure that also determines the model transitions. By removing the measurement-driven terms, the filtering process of [7] can be reduced to a continuous-time IMM extrapolation formula. The stochastic process of [7] is more or less classical jump Markov process, while our model is a continuous-time stochastic process that may jump around state spaces with different dimensions. In other words, the uniqueness of our model resides in the fact that the hybrid system state $(x_t, m_t)$ jumps from a space $E_m \times \{m\}$ to another space $E_{m'} \times \{m'\}$, rather than jumps within one given Euclidean space, with the continuous time parameter.

## 3 IMM Algorithm

First we consider the extrapolation step as described in [14]. To do so, we need to define a semi-group of linear functionals $\mathcal{T}_h$ on the space $\mathcal{C}$ of all the real-valued bounded continuous functions $\phi$ on the hybrid space $E$ by, for each $t \in [0, \infty)$ and $h \geq 0$, $\mathcal{T}_h\phi(x,m) = \mathbb{E}(\phi(x_{t+h}, m_{t+h}) \mid x_t = x, m_t = m)$. Since $(x_t, m_t)$ is a time-homogeneous Markov process, the definition does not depend on *t*. Then the *infinitesimal generator* $\mathcal{A}$ of $\mathcal{T}_h$ can be defined as

$$\begin{aligned}\mathcal{A}\phi(x,m) &= \lim_{h \downarrow 0} h^{-1}(\mathcal{T}_h\phi(x,m) - \phi(x,m)) \\ &= \frac{\partial}{\partial x}\phi(x,m)(A_m x + b_m) + \frac{1}{2}\text{trace}\left(\frac{\partial^2}{\partial x^2}\phi(x,m)Q_m\right) \\ &\quad + \sum_{m'=1}^{M} c_{mm'} \int_{E_{m'}} \phi(x',m')\mathcal{G}(dx'; F_m^{m'}x + g_m^{m'}, V_m^{m'})\end{aligned} \tag{6}$$

More precisely, when the limit, $\lim_{h \downarrow 0} h^{-1}(\mathcal{T}_h\phi - \phi)$, exists in the sup-norm of $\mathcal{C}$, we say the functional $\phi$ belongs to the domain of $\mathcal{A}$, i.e., $\phi \in \text{Dom}(\mathcal{A})$, and the last expression of eqn. (6) is uniquely implied[5] by eqns. (1) and (2). Then, for any $\phi \in \text{Dom}(\mathcal{A})$, we have [11]

$$\begin{aligned}\mathbb{E}(\phi(x_{t+h}, m_{t+h}) \mid (x_t, m_t)) &= \phi(x_t, m_t) \\ &+ \mathbb{E}\left(\int_t^{t+h} \mathcal{A}\phi(x_\tau, m_\tau)d\tau \middle| (x_t, m_t)\right)\end{aligned} \tag{7}$$

Taking the (unconditional) expectation of both sides of (7), under a condition that allows us to interchange the expectation and the time-integral, we have

$$\begin{aligned}\mathbb{E}(\phi(x_{t+h}, m_{t+h})) &= \mathbb{E}(\phi(x_t, m_t)) \\ &+ \int_t^{t+h} \mathbb{E}(\mathcal{A}\phi(x_\tau, m_\tau))d\tau\end{aligned} \tag{8}$$

or $\frac{d}{dt}\mathbb{E}(\phi(x_t, m_t)) = \mathbb{E}(\mathcal{A}\phi(x_t, m_t))$.

As in [14], let us define $\overline{p}_{mk}(t) = \text{Prob}\{m_t = m \mid y_1, ..., y_k\}$, $\overline{x}_{mk}(t) = \mathbb{E}(x_t \mid m_t = m, y_1, ..., y_k)\overline{p}_{mk}(t)$, and $\overline{S}_{mk}(t) = \mathbb{E}(x_t x_t^T \mid m_t = m, y_1, ..., y_k)\overline{p}_{mk}(t)$, for each $m = 1, ..., M$. Then it follows from (1), (2) and (8) that, for each $t \in [t_k \;\; t_{k+1}]$, with $C$ defined as the $M \times M$ matrix whose $(i, j)$ element is $c_{ij}$,

$$[\overline{p}_{1k}(t) \; ... \; \overline{p}_{Mk}(t)] = [\hat{p}_{1k} \; ... \; \hat{p}_{Mk}]\exp(C(t - t_k)) \tag{9}$$

$$\begin{aligned}\frac{d}{dt}\overline{x}_{mk}(t) &= A_m\overline{x}_{mk}(t) + b_m\overline{p}_{mk}(t) \\ &+ \sum_{m'=1}^{M} c_{m'm}(F_{m'}^{m}\overline{x}_{m'k}(t) + g_{m'}^{m}\overline{p}_{m'k}(t))\end{aligned} \tag{10}$$

and

$$\begin{aligned}\frac{d}{dt}\overline{S}_{mk}(t) &= A_m\overline{S}_{mk}(t) + \overline{S}_{mk}(t)A_m^T \\ &+ b_m\overline{x}_{mk}(t)^T + \overline{x}_{mk}(t)b_m^T + Q_m\overline{p}_{mk}(t) \\ &+ \sum_{m'=1}^{M} c_{m'm}\Big(F_{m'}^{m}\overline{S}_{m'k}(t)(F_{m'}^{m})^T + F_{m'}^{m}\overline{x}_{m'k}(t)(g_{m'}^{m})^T \\ &\quad + g_{m'}^{m}\overline{x}_{m'k}(t)^T(F_{m'}^{m})^T + (g_{m'}^{m}(g_{m'}^{m})^T + V_m^{m'})\overline{p}_{m'k}(t)\Big)\end{aligned} \tag{11}$$

The initial conditions for (10) and (11) are given as $\overline{x}_{mk}(t_k) = \hat{x}_{mk}\hat{p}_{mk}$ and $\overline{S}_{mk}(t_k) = (\hat{V}_{mk} + \hat{x}_{mk}\hat{x}_{mk}^T)\hat{p}_{mk}$.

Eqn. (9) is a well known formula, while the derivation of eqns. (10) and (11) are given in [14].

For each $t \in [t_k \;\; t_{k+1}]$, let $\varXi_t = (\overline{p}_{mk}(t), \overline{x}_{mk}(t), \overline{S}_{mk}(t))_{m=1}^{M}$ and let $\varphi$ be the function that arranges all the elements in $\varXi_t$ into a vector in the *N*-dimensional Euclidean space [6], with

[5] See Appendix A of [14] for the derivation of (5) from (1) and (2).

[6] We only need the values for the upper triangle elements for each symmetric matrix $\overline{S}_{km}(t)$.

$N=\sum_{m=1}^{M}(1+\dim(E_m)+\dim(E_m)(\dim(E_m)+1)/2)$. Then, since all the equations (9) to (11) are linear ordinary differential equations, we have ([14])

$$\Xi_t=\varphi^{-1}(\exp(Dt)\varphi(\Xi_0)) \tag{12}$$

where $D$ is an $N\times N$ matrix uniquely defined by eqns. (9) to (11), and can be calculated by any one of the well-known effective numerical methods.

Furthermore, if we assume $\overline{p}_{mk}(t)>0$ for any $m=1,...,M$ and $t\in[t_k \quad t_{k+1}]$, it follows from (9) to (11) that

$$\begin{aligned}&\frac{d}{dt}\tilde{V}_{mk}(t)=A_m\tilde{V}_{mk}(t)+\overline{V}_{mk}(t)A_m^T+Q_m\overline{p}_{mk}(t)\\&\quad+\sum_{m'=1}^{M}c_{m'm}\left(F_{m'}^m\tilde{V}_{m'}(t)(F_{m'}^m)^T+\overline{p}_{m'k}(t)(V_{m'}^m+\Delta_{m'}^m(t)\Delta_{m'}^m(t)^T)\right)\end{aligned} \tag{13}$$

with

$$\tilde{V}_{mk}(t)\overset{\text{def}}{=}\mathbb{E}\left(\left(x_t-\frac{\overline{x}_{mk}(t)}{\overline{p}_{mk}(t)}\right)\left(x_t-\frac{\overline{x}_{mk}(t)}{\overline{p}_{mk}(t)}\right)^T\middle| m_t=m,y_1,...,y_k\right)\overline{p}_{mk}(t) \tag{14}$$

and

$$\begin{aligned}\Delta_{m'}^m(t)&\overset{\text{def}}{=}\overline{p}_{mk}(t)^{-1}\overline{x}_{mk}(t)-\overline{p}_{m'k}(t)^{-1}F_{m'}^m\overline{x}_{m'k}(t)-g_{m'}^m\\&=\overline{x}_{mk}(t)-(F_{m'}^m\overline{x}_{m'k}(t)+g_{m'}^m)\end{aligned} \tag{15}$$

We should note that, in (13) to (15), we have $V_m^m=0$ and $\Delta_m^m=0$, for each $m$.

The IMM update step, which would precede each extrapolation step described above, can be performed by the standard IMM update formula. Namely, for each $m=1,...,M$, assuming $\overline{p}_{m(k-1)}(t_k)>0$, we have

$$\hat{x}_{mk}=\frac{\overline{x}_{m(k-1)}(t_k)}{\overline{p}_{m(k-1)}(t_k)}+K_{mk}\left(y_k-H_{mk}\frac{\overline{x}_{m(k-1)}(t_k)}{\overline{p}_{m(k-1)}(t_k)}\right) \tag{16}$$

$$\hat{V}_{mk}=(I-K_{mk}H_{mk})\overline{V}_{mk} \tag{17}$$

where

$$\overline{V}_{mk}=\frac{\overline{S}_{m(k-1)}(t_k)}{\overline{p}_{m(k-1)}(t_k)}-\left(\frac{\overline{x}_{m(k-1)}(t_k)}{\overline{p}_{m(k-1)}(t_k)}\right)\left(\frac{\overline{x}_{m(k-1)}(t_k)}{\overline{p}_{m(k-1)}(t_k)}\right)^T \tag{18}$$

and

$$K_{mk}=\overline{V}_{mk}H_{mk}^T\mathcal{S}_{mk}^{-1} \tag{19}$$

with

$$\mathcal{S}_{mk}=H_{mk}\overline{V}_{mk}H_{mk}^T+R_k \tag{20}$$

$$\hat{p}_{mk}=L_{mk}/\sum_{m'=1}^{M}L_{m'k} \tag{21}$$

and

$$L_{mk}=\frac{\overline{p}_{m(k-1)}(t_k)}{\sqrt{\det(2\pi\mathcal{S}_{mk})}}\exp\left(\frac{1}{2}\left\|y_k \quad \frac{\overline{x}_{m(k-1)}(t_k)}{\overline{p}_{m(k-1)}(t_k)}\right\|_{\mathcal{S}_{mk}^{-1}}^2\right) \tag{22}$$ [7]

The matrix $H_{mk}$ in (16) – (20) is the observation matrix and $R_k$ is the covariance matrix of the observation noise $\eta_k$, both in eqn. (4).

*Remark 2*: In order to derive the first and the second moments through eqns. (10) and (11), to be precise, we need an extra step, since, for example, $\phi(x,m)=x_i$ if $m=m'$, 0 otherwise, does not define a bounded functional $\phi$. In order to justify the use of eqns. (6) to (8), we may need to consider a series of stopped processes, each bounded by a compact set $\{(x,m)\,|\,\|x\|\le k\}$ for each integer *k*, and to apply Dynkin's lemma to obtain the desired result as a limit, as is done in [12] and [13].

*Remark 3*: In a sense, a key to develop a very simple solution in the form of the linear ordinary differential eqn. (12) is our use of the particular form of the first and the second moments, $\overline{x}_{mk}(t)$ and $\overline{S}_{mk}(t)$, rather than a usual choice of conditional mean and covariance, $\mathbb{E}(x_t\,|\,m_t,y_1,...,y_k)$ and $\mathbb{E}(x_tx_t^T\,|\,m_t,y_1,...,y_k)-\mathbb{E}(x_t\,|\,m_t,y_1,...,y_k)\mathbb{E}(x_t\,|\,m_t,y_1,...,y_k)^T$. To the best of our knowledge, this fact was shown in [8] for the first time, and has been expanded to a general multiple-model, affine-Gaussian dynamics and jumps in [14]. As mentioned before, in contrast, the continuous-time IMM formulation of [7], devised 30 some years ago, is based on a form of classical jump Markov stochastic process.

# 4 A Simple Financial Market Model

As a financial market model, we will use a simple multiple-model switching system, as in [18-20]. We have three models (i.e., $M=3$), (i) the "up" ("bull"), (ii) "steady," and (iii) "down" ("bear") models. Generally, by "*u*," we mean the "price" in an appropriate sense, usually some sort of average price, and by "*v*," its time derivative. The three models are defined as follows:

(i) Up (Bull) Model ($m=1$) is based on a biased Ornstein-Uhlenbeck process, defined by the affine stochastic differential equation,

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[7] $\|x\|_A\overset{\text{def}}{=}\sqrt{x^TAx}$

$$\begin{cases} du_t = v_t dt \\ dv_t = -\beta_1(v_t - \bar{v}_1)dt + \sqrt{q_1}dw_t \end{cases} \tag{23}$$

with unit-intensity Wiener process $w_t$, and three strictly positive parameters, $(\bar{v}_1, \beta_1, q_1)$.

(ii) Steady Model ( $m = 2$ ) is a one-dimensional stationary stochastic process defined by

$$du_t = -\beta_0(u_t - \bar{u}_0)dt + \sqrt{q_0}dw'_t \tag{24}$$

with unit-intensity Wiener process $w'_t$, and three strictly position parameters, $(\bar{u}_0, \beta_0, q_0)$.

(iii) Down (Bear) Model ( $m = 3$ ) is another biased Ornstein-Uhlenbeck process defined by

$$\begin{cases} du_t = v_t dt \\ dv_t = -\beta_1(v_t + \bar{v}_1)dt + \sqrt{q_1}dw''_t \end{cases} \tag{25}$$

also with unit-intensity Wiener process $w''_t$. We may have a different set of parameters. But, for simplicity, we use the same set of parameters of Model 1.

Thus we have $A_1 = A_3 = \begin{bmatrix} 0 & 1 \\ 0 & -\beta_1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 0 \\ \beta_1\bar{v}_1 \end{bmatrix} = -b_3$, $B_1 = B_3 = \begin{bmatrix} 0 \\ \sqrt{q_1} \end{bmatrix}$, $Q_1 = Q_3 = \begin{bmatrix} 0 & 0 \\ 0 & q_1 \end{bmatrix}$, $A_2 = [-\beta_0]$, $b_2 = [\beta_0\bar{u}_0]$, $B_2 = [\sqrt{q_0}]$, and $Q_2 = [q_2]$ with $E_1 = E_3 = (-\infty,\infty)^2$ and $E_2 = (-\infty,\infty)$. Assuming the symmetry, the transition probabilities of eqn. (1) is defined by

$$C = \left(\left(c_{mm'}\right)_{m=1}^{M}\right)_{m'}^{M} = \begin{bmatrix} -c_1 & c_1 & 0 \\ c_2/2 & -c_2 & c_2/2 \\ 0 & c_1 & -c_1 \end{bmatrix} \tag{26}$$

with two parameters, $c_1 > 0$ and $c_2 > 0$. $F_1^2 = F_3^2 = [1 \quad 0]$, $g_1^2 = g_3^2 = V_1^2 = V_1^3 = [0]$, $F_2^1 = F_3^1 = [1 \quad 0]^T$, $g_2^1 = [0 \quad \bar{v}_1]^T = -g_3^1$, and $V_2^1 = V_3^1 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{\sigma}_1^2 \end{bmatrix}$, with $q_1 = 2\beta_1\sigma_1^2$.

Then we can write eqn. (11) explicitly as

$$\frac{d}{dt}\Xi(t) = \begin{bmatrix} D_{11} & 0 & 0 \\ D_{21} & D_{22} & 0 \\ D_{31} & D_{32} & D_{33} \end{bmatrix}\Xi(t) \tag{27}$$

with $\Xi(t) = \left[\bar{p}_{1k}(t) \quad \bar{p}_{2k}(t) \quad p_{3k}(t) \quad \bar{x}_k(t)^T \quad \tilde{S}_k(t)^T\right]^T$, where $\bar{x}_k(t) = \left[\bar{x}_{1k}(t)^T \quad \bar{x}_{2k}(t)^T \quad \bar{x}_{3k}(t)^T\right]^T$, $\tilde{S}_k(t) = \left[\tilde{S}_{1k}(t)^T \quad \tilde{S}_{2k}(t)^T \quad \tilde{S}_{3k}(t)^T\right]^T$ (with the vector representations $\tilde{S}_k$ and $\tilde{S}_{mk}$ for the matrices $\bar{S}_k$ and $\bar{S}_{mk}$),

$$D_{11} = \begin{bmatrix} -c_1 & c_2/2 & 0 \\ c_1 & -c_2 & c_1 \\ 0 & c_2/2 & -c_1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 0 & 0 \\ \beta\bar{v} & c_2\bar{v}_1/2 & 0 \\ 0 & \beta_0\bar{u}_0 & 0 \\ 0 & 0 & 0 \\ 0 & -c_2\bar{v}_1/2 & -\beta\bar{v} \end{bmatrix},$$

$$D_{22} = \begin{bmatrix} -c_1 & 1 & c_2/2 & 0 & 0 \\ 0 & -\beta_1 - c_1 & 0 & 0 & 0 \\ c_1 & 0 & -\beta_0 - c_2 & c_1 & 0 \\ 0 & 0 & c_2/2 & -c_1 & 1 \\ 0 & 0 & 0 & 0 & -\beta_1 - c_1 \end{bmatrix},$$

$$D_{31} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_1 & c_2(\bar{v}_1^2 + \bar{\sigma}_v^2)/2 & 0 \\ 0 & q_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_2(\bar{v}_1^2 + \bar{\sigma}_v^2)/2 & q_1 \end{bmatrix},$$

$$D_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \beta\bar{v} & 0 & c_2\bar{v}_1/2 & 0 & 0 \\ 0 & 2\beta\bar{v} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_2\bar{v}_1/2 & -\beta\bar{v}_1 & 0 \\ 0 & 0 & 0 & 0 & -2\beta\bar{v}_1 \end{bmatrix}, \text{ and}$$

$$D_{33} = \begin{bmatrix} -c_1 & 2 & 0 & c_2/2 & 0 & 0 & 0 \\ 0 & -\beta_1 - c_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\beta - c_1 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & -c_2/2 & c_1 & 0 & 0 \\ 0 & 0 & 0 & c_2/2 & -c_1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_1 - c_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\beta_1 - c_1 \end{bmatrix}$$

The initial condition is given by the initial model probabilities $p_{m0} \equiv 1/3$, for $m = 1,2,3$, and using the first measurement at time $t_1 = t_0$, by $\bar{x}_{11}(t_1) = [y_1 \ \bar{v}_1]^T p_{10}$, $\bar{x}_{21}(t_1) = [y_1]p_{20}$, $\bar{x}_{31}(t_1) = [y_1 \ \text{-}\bar{v}_1]^T p_{30}$, $\bar{S}_{11}(t_1) = \text{diag}(R_1, \bar{\sigma}_{v_1}^2)p_{10} - \bar{x}_{11}(t_1)\bar{x}_{11}(t_1)^T / p_{10}$,

$\bar{S}_{21}(t_1) = R_1 p_{20} + \bar{x}_{21}(t_1)^2 / p_{20}$ , and $\bar{S}_{31}(t_1) = \text{diag}(R_1, \bar{\sigma}_{v_1}^2) p_{30} + \bar{x}_{31}(t_1)\bar{x}_{31}(t_1)^T / p_{30}$ .

The measurement matrices are given $H_{1k} = H_{3k} = [1\ 0]$ and $H_{2k} = [1]$, for all $k = 1, 2, 3, \ldots$

# 5 Numerical Examples

We applied the dynamic model described in the previous section to the S&P market. The historical data from 1980 to 2014 was tested [15]. Specifically, the close prices were used as the measurements and the three dynamic models: "up (bull)", "steady", and "down (bear)" as described in the previous section were used to assess the market condition. Figure 1 shows the S&P historical data and the monthly returns from 1980 to 2014. We randomly selected one daily, one weekly, and one monthly data sets, each with 100 data points to test the algorithm respectively.

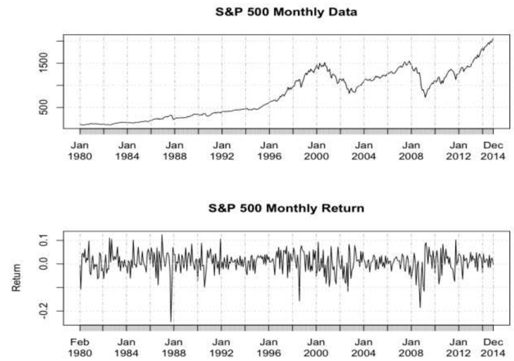


Figure 1. Monthly S&P form 1980 to 2014

In each test, the resulting estimated probabilities of the three models are used to make the asset allocation decisions. Similar to other popular technical indicators such as stochastic oscillator (SO) or relative strength index (RSI) [16], the resulting "IMM" model probabilities are considered as a momentum indicator. This new indicator attempts to determine the potential market overbought or oversold conditions. For example, when the "up" model probability is the highest one among the three and is above a certain threshold, it may indicate an overbought condition, and when the "down" model probability is the highest one and is above a certain threshold, it may indicate an oversold condition.

With the IMM indicator, we dynamically allocate the asset and make trading decisions accordingly. A simple strategy is to short (sell) the S&P futures when the "up" probability is the highest one (overbought) and to long (buy) when the "down" probability is the highest one. We may also want to close our positions and sit on sideline when the market is uncertain ("steady" mode probability is the highest). However, this "contrarian" approach may not work well in the market with a strong up or down trend. To mitigate this risk, when the IMM "up" or "down" probabilities are in extreme values (say, > 0.95) which indicates a potential strong trend, the decision rule mentioned above will reverse to follow the market directions.

With the above simple asset allocation rules based on the IMM indicator, we conduct simulation and test its performance on the three randomly selected S&P data sets. We also compare their performances with the naïve buy-and-hold policy.

**I. Daily Data**

Figure 2 shows a randomly selected daily S&P closing prices and returns over a 100 days period. The daily returns represent the daily equity percentage changes of the buy-and-hold strategy. Figure 3 shows the probability trajectories of the three models estimated by the IMM algorithm.

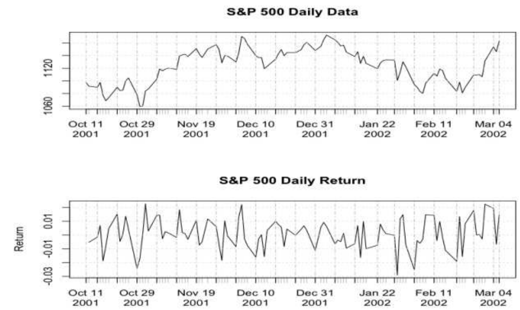


Figure 2. S&P Daily Data – 100 Days

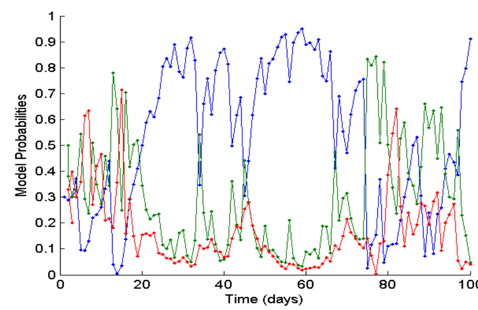


Figure 3. IMM Model Probabilities – S&P Daily Data

The corresponding trading decisions of the IMM dynamic asset allocation (IMM-DAA) strategy and its daily returns are shown in Figure 4. In the figure, decision "1" represents a long position, "-1" represents a short position, and "0" represents no position. Figure 5 compares the equity curve over the 100 days period for the DAA strategy and the buy-and-hold (BH) policy. As seen from the figure, DAA performs significantly better than the BH strategy with only a few trading actions - a total of around 20 over the 100 days period. At the end of the 100 days period, the cumulative return for BH is under 6% while DAA's return is almost 14%. Note that the maximum drawdown[8] of the BH policy is approximately 7% while the maximum drawdown of the DAA is only about 5%.

[8] Drawdown is defined as the peak-to-trough decline during a specific period of an investment. A drawdown is usually quoted as the percentage between the peak and the trough [16].

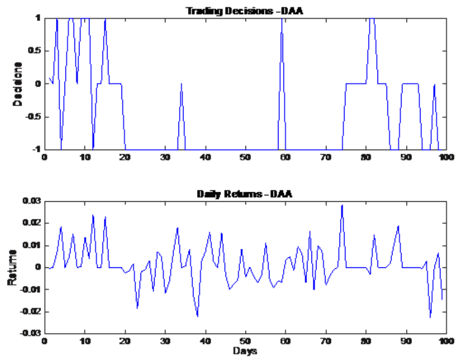


Figure 4. IMM-DAA Trading Decisions and Daily Returns

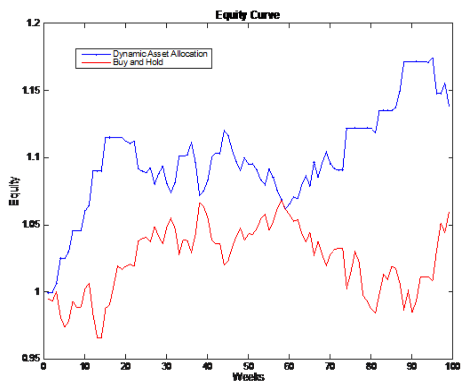


Figure 5. Equity Curves – Buy-and-Hold vs. DAA

## II. Weekly and Monthly Data

Figures 6-9 show the results corresponding to the weekly and monthly data. Note that the decision rules based on the IMM indicators are exactly the same for the three data sets. The only parameter that has to be changed to adapt to the different time intervals is the variance of the process noise in the IMM model. Based on the Markov property, the standard deviation of the process noise (volatility) is proportion to the square root of the time difference between two subsequent observations.

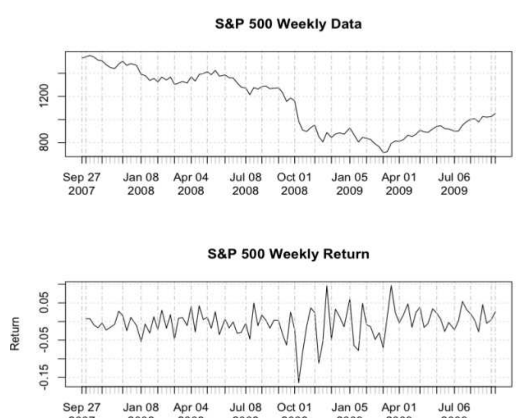


Figure 6. S&P Weekly Data

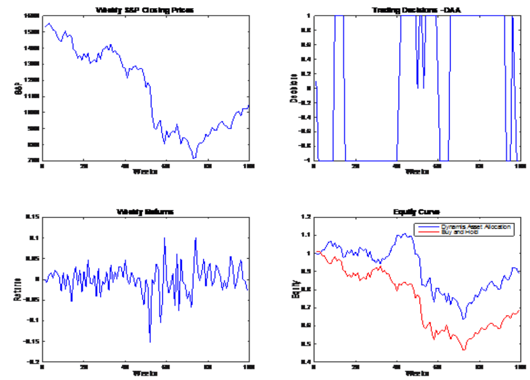


Figure 7. Trading Performance - Weekly Data

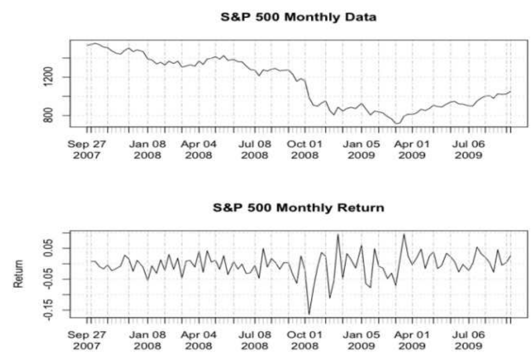


Figure 8. S&P Monthly Data

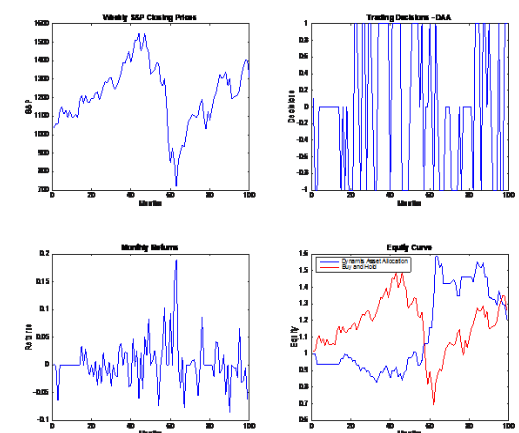


Figure 9. Trading Performance – Monthly Data

As shown in the figures, DAA either performs better than or close to the BH with significantly lower drawdown. For example, Figure 7 shows that while BH loses about 31% of the equity over the 100 weeks period with a maximum drawdown of about 54%, DAA only loses 11% with a maximum drawdown of 43% over the same period. Similarly, Figure 9 shows that while BH earns about 26% of the equity over a 100 months period with a maximum drawdown of about 54%, DAA earns a slightly less return of 21% over the same period with a significantly smaller drawdown of only 24%.

Table 1 summarizes the performance results for the three randomly selected data sets. In the table, an industry-standard performance indicator called the "Sharpe ratio"[9] is also presented for performance comparison. Higher Sharpe ratio indicates a better risk-adjusted return. It is clear from the table that the IMM based DAA (IMM-DAA) is an effective and promising asset allocation method.

Table 1. Performance Comparison

| | Rate of Return | Maximum Drawdown | Sharpe Ratio |
|---|---|---|---|
| Daily - BH | 5.95% | 7.88% | 0.878 |
| Daily- DAA | 13.86% | 5.40% | 2.307 |
| Weekly - BH | -31.26% | 54.14% | -0.615 |
| Weekly- DAA | -10.88% | 42.60% | -0.134 |
| Monthly - BH | 26.08% | 53.54% | 0.192 |
| Monthly - DAA | 21.00% | 23.62% | 0.160 |

# 6 Conclusion

In this paper, we described a continuous-time, discrete-observation, Interacting Multiple Model (IMM) algorithm, based on the continuous-time IMM extrapolation developed in [14], and applied it to financial market dynamic modeling. We modeled the system by a continuous-time, jump Markov process and estimated the system state based on a sequence of discrete-time, linear-Gaussian observations. We utilized a rather naïve switching process with multiple linear stochastic system models to represent the S&P market dynamic model. The resulting IMM model probabilities are served as a momentum indicator to make the dynamic asset allocation decisions (DAA). We tested the resulting IMM-DAA strategies on several randomly selected S&P data sets of various time intervals. The results showed that the new strategy is either comparable or significantly outperforms the naïve buy-and-hold policy with much less risk.

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[9] The Sharpe ratio is a measure for calculating risk-adjusted return. It is the average return earned in excess of the risk-free rate over the return volatility (standard deviation).


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