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An Effective Modeling Framework for Equality-Constrained Dynamic Systems

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***Abstract*—Due to physical laws or mathematical properties the parameters and/or the states of some dynamic systems satisfy certain constraints, and exploitation of such constraints generally is expected to produce more accurate system models. This paper is concerned with modeling of the dynamic systems with equality constraints. An effective framework of the constrained dynamics modeling is proposed by which the equality constraints and the auxiliary (unconstrained) dynamics are optimally fused. In particular, modeling of linear equality constrained dynamic systems and quadratic equality constrained dynamic systems is systematically investigated. Meanwhile, the effects of the auxiliary dynamics on the constructed dynamic model are analyzed. Finally, the proposed modeling is assessed on a benchmark scenario of road-based vehicle tracking.**

***Index Terms*—Dynamics modeling, equality constraint, constrained optimization.**

I. Introduction

Constrained dynamic systems occur frequently in which the state components (are required to) satisfy certain constraints arising from physical laws or mathematical properties [1]. For instance, in electric circuits, voltages and currents obey the Kirchhoff's laws [2]; in spacecraft attitude determination, quaternion of rotation is subject to unit-norm constraint [3], [4]; in traffic control, land-based vehicles are road-constrained [5], [6], and civil aircrafts are required to fly within the preset flight envelop [7]. Such constraints are of various forms including equality/inequality constraint [8], set constraint [9], probability constraint [7], performance constraint [10], hard/soft constraint, etc. Such constraints contain valuable information about the system state and they should be taken into account in system analysis and many other problems.

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While there are many research topics concerning the constrained problems, we focus in this paper on the dynamic systems with equality constraints and address the problem of the constrained dynamics modeling. The importance of the constrained dynamics modeling for system analysis cannot be overstated because the constructed model has such potentially useful applications as control, filtering, dynamics analysis, and system identification. Notice that constraints are often known a priori and thus they should be incorporated into the dynamic model, which requires that the state according to the model always satisfies the constraints automatically. In practice, however, directly constructing such a constrained dynamic model is often very difficult, especially for complex systems, and the literature on this topic is scarce. [11] and [12] focused on time-invariant and homogeneous linear equality constraints and developed a constrained dynamic model called projection system. [13] presented necessary conditions on both the dynamics and the process noise for the linear equality constrained state. [14] developed a reduced state space model for a linear equality constrained system using the null space decomposition. By using the direct elimination technique, [15] and [16] discussed how to design a dependent constrained subsystem from an independent unconstrained subsystem when a desired model class is given. Then the design is specifically applied for constrained target motion modeling on a straight line [17] and a circular track [18]. Recently, taking into account the randomness of the state and using the Gram-Schmidt decomposition technique, [19] derived a linear equality constrained dynamic model which is optimal in a certain sense. Additionally, almost all of these studies deal with the case of linear equality constraints, but to our knowledge, modeling for dynamic systems with more general constraints (e.g., nonlinear equality constraint) is rare in the literature.

In contrast, the problem of constrained estimation has received much attention in these years and several equality constrained estimation methods have been proposed (see [20], [21] for surveys). Essentially, the constrained state estimation is not different from the conventional estimation except that it is based on the constrained dynamic model. It has been proved in [19], [22] that using the linear equality constrained dynamic model (LECDM), the estimation result produced by the conventional filter (e.g., the Kalman filter) will automatically satisfy the constraints so that no post treatment of enforcing the constraints is needed. However, most of the current constrained estimation methods bypass the constrained dynamics modeling problem. Instead, they calculate the posterior estimate or distribution based on an *unconstrained* dynamic model first and then impose the constraints on the posterior estimate or distribution to yield a constrained estimate. Their representatives are model reduction, pseudo measurement and estimate projection methods. The model reduction method reparameterizes the constrained system so that the linear equality constraint is naturally satisfied [14]. Although this method has merits in numerical stability and computational complexity, physical meanings of the state components after reparameterization are often lost. More importantly, it is difficult to extend this method to the systems with nonlinear equality constraints. This method is beyond the scope of this paper, while the other two constrained estimation methods will be discussed briefly for comparison purposes.

The pseudo measurement (PM) method augments the original measurement by a fictitious measurement without noise to account for the effect of the equality constraints, and then a conventional filter is applied [23], [24]. This method has a wide range of applications in target tracking [25], economics [24], [26], [27], etc. However, it may suffer from numerical instability because of the singularity of the error covariance matrix of the augmented measurement [20], [28], [29]. More remarkably, the equality constraint differs from the noise-free measurement significantly, so simply treating an equality constraint as a measurement and applying the traditional filtering without modification is not rigorous, as clearly and convincingly justified in [19]. In addition, for nonlinear equality constraints, the state estimates produced by this method do not necessarily obey the constraints.

The estimate projection (EP) method projects the unconstrained estimate onto the constraint surface by applying classical constrained optimization techniques, which is one of the most popular constrained estimation methods in applications. [30] proposed this method first, based on which the state estimation problem for linear equality constrained systems was tackled. Then, [28] and [13] showed that some constrained estimation methods (e.g., pseudo measurement) are mathematically equivalent to this method under certain conditions. Later, [1] and [31] devoted attention to the method and extended it to the nonlinear equality constraint case, and especially [1] argued to perform the projection procedure twice for pursuiting higher estimation accuracy. Nevertheless, this method has at least two debatable issues. First, they cannot guarantee a true optimality [14], that is, the projected point being close to the unconstrained estimate does not imply that it is close to the true constrained state. Second, they produce the constrained estimate by applying the constraint only to the updated estimate or conditional distribution, rather than to the whole system, and so the effects of the constraints on the prior (or predicted) distribution of the state are not sufficiently considered [19].

In this paper, the main contribution is to propose an effective modeling framework for the equality-constrained systems. While it is often difficult to formulate the exact constrained dynamic model directly, an unconstrained dynamic model that approximates the state evolution of the constrained systems sometimes is readily available in which the state does not necessarily obey the constraints. By this approach, the constructed dynamic model optimally fuses the constraint information and the unconstrained dynamics. The modeling problem of the dynamic systems with two typical constraints—linear equality constraints (LEC) and quadratic equality constraints (QEC)—is solved. For the LEC case, the constrained dynamic model is formulated analytically and its form reduces to the LECDM presented in [11], [12], [19]. Compared with those using state space decomposition techniques, our method here is more intuitive and easy to implement. Through effectively solving the quadratic equality constrained optimization problem, an expression describing the evolution of the state with QEC is also investigated. Via numerical simulations on tracking road-based vehicles, we evaluate the effectiveness of our constrained dynamics modeling method.

This paper is organized as follows. The modeling problem for constrained systems is posed in Section II. In the next section, a criterion for optimally fusing the constraint information and the unconstrained dynamics is proposed, under which a unified modeling framework for the constrained dynamics is presented. The effectiveness of the approach is demonstrated in Section IV by an example of road-based vehicle tracking. The last section draws some conclusions.

II. Problem Formulation

Consider a dynamic system whose state vector $x_k \in \mathbb{R}^n$ obeys the following constraint

$$x_k \in \mathcal{D}_k, \quad k = 0, 1, \cdots \tag{1}$$

where $\mathcal{D}_k$ stands for a known constraint subset at time k. In particular, for a general equality constraint, $\mathcal{D}_k$ is defined as

$$\mathcal{D}_k = \{x_k : c_k(x_k) = 0\} \tag{2}$$

where $c_k(\cdot)$ is a given vector-valued function. The constrained state x_k is observed by

$$z_k = h_k(x_k) + v_k \tag{3}$$

where $h_k(\cdot)$ is the measurement function, z_k is the measurement, and the measurement noise v_k is white Gaussian with zero mean and covariance R_k.

The exact dynamic model of the constrained system must be consistent with the constraint. This means that the state evolving as the exact model always automatically satisfies the constraints. However, the exact model is often unknown or too difficult to build directly, especially for complicated constraint cases. In contrast, an unconstrained dynamic model which approximates the actual system is often easily available. Such an unconstrained dynamic model is helpful for the constrained dynamics modeling and thus it is called auxiliary dynamics. Suppose that the auxiliary dynamics of the constrained system is known and given by

$$x_{k+1}^a = f_k(x_k^a, u_k, w_k^a), \quad k = 0, 1, 2, ... \tag{4}$$

where x_k^a stands for the state of the auxiliary system, and $f_k(\cdot)$ is a given vector-valued function. The process noise w_k^a is assumed to be zero-mean and white Gaussian with $\mathrm{cov}(w_k^a) = Q_k^a$. The initial state x_0^a has the Gaussian distribution with mean $\bar{x}_0^a$ and covariance Σ_0^a. The sequences $\{w_k^a\}$ and $\{v_k\}$ are assumed to be mutually independent as well as independent to fx_0^a. Note that the auxiliary system (4) is of a general form and its state x_k^a does not necessarily satisfy the constraint (2).

Our goal is to propose a unified modeling approach for the equality constrained dynamics. The constraints (2) and the auxiliary system (4) are two complementary forms of our prior knowledge of the evolution behavior of the system state, and thus the constrained dynamic model is constructed by fusing these two pieces of prior information optimally in a certain sense.

III. Modeling for Dynamic Systems with Equality Constraints

A. *Modeling Criterion for Constrained State Evolution*

Rather than to design the structure and parameters of the constrained dynamics directly, we turn to derive the constrained dynamic model from the given auxiliary system (4) and the known constraint (2). As stated before, the auxiliary system describes roughly the evolution of the actual constrained system—its state x^a approximates the true state x. Then, an intuitive criterion for modeling the constrained state is the nearest neighbor criterion

$$\min_{x \in \mathcal{D}} d(x, x^a) \tag{5}$$

where $d(x, x^a)$ refers to the distance between x and x^a in a general sense. That is, the state vector x to be chosen is the point in the constraint subset which is closest to x^a. Once the optimization result x of (5) for each x^a is available, the distribution of the constrained state x is obtained. Herein, choosing a reasonable distance metric d and solving the optimization problem (5) are two fundamental problems to construct the constrained state x. This is discussed next.

There exists numerous well-defined distance metrics, such as the commonly used ℓ_p-norm ($p \geq 1$). By different distance metrics, the values for describing the closeness of two points in a multi-dimensional space differ quantitatively in general [32]. On the other hand, solving the optimization problem (5) using different distance metrics may also result in different optimal x. Thus, a basic prerequisite for modeling the constrained state is the choice of an appropriate distance metric. By comparison, the Euclidean distance (i.e., ℓ_2-norm) is widely used for its simplicity and clear physical meaning, which is also chosen in this work. Through combining the auxiliary system (4), the proposed modeling criterion for the constrained evolution is described as follows

$$x_k = \arg\min_{x \in \mathcal{D}_k} \|x - x_k^a\|_W^2 \tag{6}$$

$$= \begin{cases} \arg\min_{x \in \mathcal{D}_0} \|x - x_0^a\|_W^2, \quad k = 0 \\ \arg\min_{x \in \mathcal{D}_k} \left\|x - f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}^a)\right\|_W^2, \\ \qquad\qquad k = 1, 2, ... \end{cases} \tag{7}$$

where $\|x\|_W = \sqrt{x^\top W x}$ is often called the W-norm and W is a user-defined symmetric and positive definite matrix. A complete description of the constrained state comprises the initial state and the state transition equation. The modeling framework for the initial constrained state and the constrained state transition is shown by the two equations in (7). Note that the state x_{k-1}^a is replaced with x_{k-1} in the second equation of (7) since the state is also constrained at time $k-1$.

Remark 1: If the given auxiliary dynamics is already consistent with the constraint; that is, if the state variable x_k^a automatically obeys the constraint (i.e., in $\mathcal{D}_k$), clearly, the constrained dynamic model to be constructed by the proposed modeling approach is the same as the auxiliary one. Besides, the special case with $\mathcal{D}_k = \mathbb{R}^n$ is trivial since the constructed model actually equals the auxiliary (unconstrained) dynamics.

Without considering the auxiliary dynamics, the modeling criterion (7) degenerates into the typical constrained least squares problem, as shown by (6). The literature has emerged on solving such a problem is vast and many methods have been proposed. Among these methods, the Lagrange multiplier method is among the most popular ones [33].

The Lagrangian of (6) with the equality constraint (2) is

$$\mathcal{L}(x, \lambda) = \|x - x_k^a\|_W^2 + \lambda^\top c_k(x), \tag{8}$$

and the constrained state is obtained by minimizing (8). Suppose that the derivative $\nabla_x c_k(x)$ of $c_k(x)$ with respect to x exists and is continuous on an open neighborhood of any point x. Then, the first order necessary conditions for the minimum are

$$\frac{\partial \mathcal{L}(x_k, \lambda)}{\partial x_k} = 0 \Rightarrow 2W(x_k - x_k^a) + \nabla_x c_k(x)\lambda = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}(x_k, \lambda)}{\partial \lambda} = 0 \Rightarrow c_k(x) = 0 \tag{10}$$

Remark 2: The constrained optimization problem (6) may have multiple solutions. In general, however, only one solution is desired for a given x_k^a in the system modeling. To this end, some strategies can be used to select a "best" one from these solutions. For example, among the solutions of (6), select the one which is nearest to x_k^a under another norm (e.g., ℓ_1-norm).

Once a one-to-one correspondence of x_k and x_k^a is obtained, the constrained dynamic model is then constructed by replacing x_k^a with the given auxiliary dynamics. Within the proposed modeling framework, we discuss next the modeling procedures for dynamic systems with linear equality constraints (LEC) and nonlinear equality constraints (NEC), respectively.

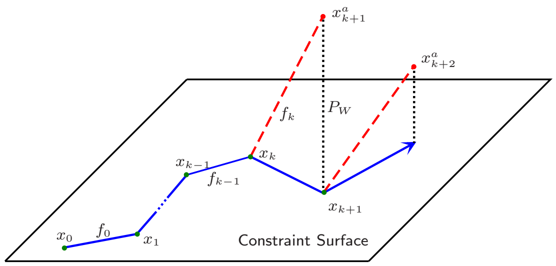


Fig. 1. Model construction of the constrained dynamic systems

B. Linear Equality Constraints

Among the various constraints, the LEC case is obviously the simplest one and relevant issues can be handled by using linear algebra. In addition, some actual systems with constraints can be approximated by a linear equality constrained model very well. For instance, in airport surveillance, the planes slide in the runway, and without considering its width, the runway can be represented by piecewise linear approximation [34].

The linear form of (2) is

$$c_k(x) = C_k x - d_k = 0 \tag{11}$$

where $C_k \in \mathbb{R}^{m\times n}$ has full row-rank with $m < n$ (the redundant constraints can be eliminated if C_k is not of full row-rank). To model the systems with the constraint (11), the state-space-decomposition based approach is commonly used [11], [14], [19]: An equivalent form of the given auxiliary dynamics is obtained first by using the Gram-Schmidt decomposition, and then let some specified random components in the form be fixed so as to be in consistent with the constraint (11).

In this paper, we deal with the construction of the constrained dynamic model within the unified modeling framework (7). Based on the LEC (11) and by calculating the first order conditions (9)–(10), the solution of (6) can be expressed as

$$x_k = P^W_k x^a_k + (I - P^W_k)x^d_k \tag{12}$$

where $P^W_k = I - W^{-1}C_k^\top(C_k W^{-1} C_k^\top)^{-1}C_k$ is the projector onto the null space of C with weight W, and $x^d_k = C_k^\top(C_k C_k^\top)^{-1}d_k$. In addition, the second-order condition is $\frac{\partial^2 \mathcal{L}(x_k,\lambda)}{\partial x_k^2} = W > 0$, which indicates that the solution (12) is the global minimum of the constrained optimization problem (6). Therefore, the linear equality constrained dynamic model (LECDM) is constructed as

$$x_k = P^W_k[f_{k-1}(x_{k-1}, u_{k-1}, w^a_{k-1})] + (I - P^W_k)x^d_k \tag{13}$$

The above dynamic model gives the evolution of the constrained state in time. As illustrated in Fig. 1, the constrained state can be realized by projecting the unconstrained state onto the constraint surface $Cx = d$. Note that in (13), W is user-defined and different W's lead to different system models, and so the constrained system modeling is affected by W. A common choice of W is the identity matrix I, and then the constrained state x is the orthogonal projection of x^a on the constraint surface. For some typical systems, however, selecting a suitable W can improve the modeling to some extent. This is discussed next.

Assume that the given auxiliary dynamics is with additive noise:

$$x^a_{k+1} = f_k(x^a_k, u_k) + w^a_k, \quad w^a_k \sim \mathcal{N}(0, Q^a_k) \tag{14}$$

For modeling such dynamics, weight W is suggested to be set to $(Q^a_k)^{-1}$, and thus the constrained dynamic model is constructed as

$$x_{k+1} = P_k[f_k(x_k, u_k) + w^a_k] + (I - P_k)x^d_{k+1} \tag{15}$$

where $P_k = I - Q^a_k C_{k+1}^\top(C_{k+1}Q^a_k C_{k+1}^\top)^{-1}C_{k+1}$ is an oblique (rather than orthogonal) projector onto the constraint surface. $P_k w^a_k$ is the process noise (or modeling error) of the newly constructed dynamic model and its associated covariance equals $P_k Q^a_k P_k^\top$. The suggested projector P_k makes model (15) have the smallest modeling error among the family of the constrained dynamic models (13) with additive noise, since for any positive definite W, the following matrix inequality always holds [19]

$$P_k Q^a_k P_k^\top \le P^W_k Q^a_k (P^W_k)^\top \tag{16}$$

where $P^W_k Q^a_k (P^W_k)^\top$ corresponds to the covariance of the process noise in (13) with additive noise.

C. Nonlinear Equality Constraints

This section considers the nonlinear equality constrained problem, which is more commonly encountered in reality. Under our unified modeling framework, the constraint optimization (6) should be solved first to obtain the mapping from x^a_k to x_k. If the solution is expressed analytically by $x_k = g(x^a_k, W)$, then the constrained dynamic model can be represented by

$$x_{k+1} = g\left(f_k(x_k, u_k, w_k), W\right) \tag{17}$$

In this model, the NEC information is embedded and any state x_k evolving according to (17) automatically satisfies the constraints.

However, in many situations, the constrained optimization result (6) has no analytical but numerical solution, and the optimization procedure depends heavily on the specified value of x^a_k. In theory, the distribution of the constrained state x is derived by solving the constrained optimization problem (6) for all x^a_k belonging to a continuous domain. Let $p(\cdot)$ denote the probability density function (pdf). Here, we use a set of support points $\{x^{a,i}_k\}$ with associated weights $\{\alpha^i\}$, $i = 1, ..., N$, to approximate $p(x^a_k)$,

$$p(x^a_k) \approx \sum_{i=1}^{N} \alpha^i \delta(x^a_k - x^{a,i}_k)$$

where $\delta(\cdot)$ stands for the Dirac delta function. Note that the point $x^{a,i}_k$ evolves from $\{x^i_{k-1}, w^{a,i}_{k-1}\}_{i=1}^N$ by the auxiliary dynamics, and $w^{a,i}_{k-1}$ is the sample of the process noise w^a_{k-1}.

TABLE I
CALCULATING THE PDF OF x_k

for $k = 1, 2, \ldots$
for $i = 1 : N$
draw $\{x_{k-1}^i, w_{k-1}^{a,i}, \alpha^i\} \sim p(x_{k-1})p(w_{k-1}^a)$
calculate $x_k^{a,i} = f_{k-1}(x_{k-1}^i, u_{k-1}) + w_{k-1}^{a,i}$
$\triangleright$ obtain x_k^i by solving Eq. (6)
end for
pdf of x_k is $p(x_k) \approx \sum_{i=1}^N \alpha^i \delta(x_k - x_k^i)$
end for

The weights $\{\alpha^i\}$ are normalized such that $\sum_{i=1}^N \alpha^i = 1$. It follows that a set of constrained points $\{x_k^i\}$ with associated weights $\{\alpha^i\}$ is obtained, which approximates the pdf of the constrained state x_k. Table I describes the steps to calculate the density of the constrained state assuming x_{k-1} and w_{k-1}^a are independent. For sampling of the point set $\{x_{k-1}^i, w_{k-1}^{a,i}, \alpha^i\}_{i=1}^N$, there are various methods in the literature, including random sampling [35] and deterministic sampling [36], [37].

In what follows, we focus on a dynamic system with a specific type of nonlinear constraint—quadratic equality constraint (QEC) in the form

$$\|L_k x_k\|^2 = 1 \tag{18}$$

where the coefficient matrix $L_k \in \mathbb{R}^{l \times n}$ has full column-rank with $l \geq n$. The QEC widely exists in spacecraft attitude estimation [4] [13], target tracking [31] [1], communication [38], etc. In order to construct a dynamic model for the system with constraint (18), as stated earlier, the following constrained optimization problem need be solved first:

$$\begin{aligned} x_k &= \arg\min_{x_k} \|x_k - x_k^a\|_W^2 \\ &\text{s.t.} \ \|L_k x_k\|^2 = 1 \end{aligned} \tag{19}$$

Fig. 2 gives a geometrical interpretation of the optimization in the case that L_k equals identity matrix I. The constraint (18) is thus a unit-circle, and the solution is the point on the unit-circle which is closest to the auxiliary state x_k^a in the W-norm sense.

Concerning the typical optimization problem (19), we provide an easy-to-implement solution by using generalized singular value decomposition (GSVD). First, using the Cholesky factorization to decompose the positive-definite matrix W as $W = G^\top G$, where G is an upper triangular matrix of full rank, we can write the Lagrange function as (for notational simplicity, we omit the time index):

$$\mathcal{L}(x, \lambda) = \|G(x - x^a)\|^2 + \lambda(\|Lx\|^2 - 1) \tag{20}$$

Further decompose matrices $G \in \mathbb{R}^{n \times n}$ and $L \in \mathbb{R}^{l \times n}$ via GSVD technique [39],

$$G = U\Sigma_G X^\top, \quad L = V\Sigma_L X^\top$$

where $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{l \times l}$ are both orthonormal matrices, $X \in \mathbb{R}^{n \times n}$ is nonsigular, $\Sigma_G = \mathrm{diag}(\sigma_{G,1}, \sigma_{G,2}, ..., \sigma_{G,n})$, $\Sigma_L = [D \ \ 0]^\top$,

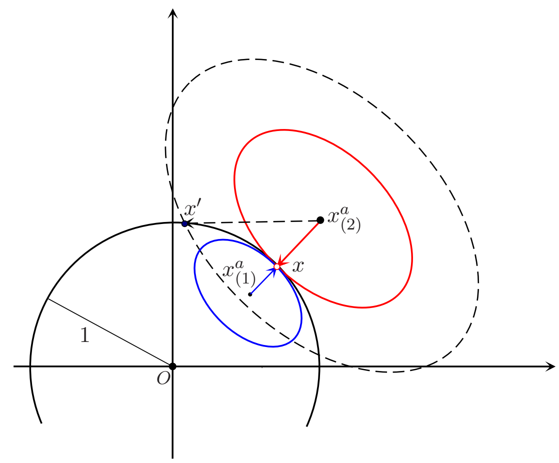


Fig. 2. Geometrical interpretation of optimization with a quadratic constraint in $\mathbb{R}^2$. Points x and x' are both the projections of auxiliary state x^a onto the unit-circle, but x is the closest one to x^a in terms of the W-norm. This figure shows two cases of such x^a.

$D = \mathrm{diag}(\sigma_{L,1}, \sigma_{L,2}, ..., \sigma_{L,n})$, $\sigma_{G,j} > 0$, and $\sigma_{L,j} > 0$, $j = 1, ..., n$. Then the first order necessary condition (9) for solving (19) can be written as

$$G^\top G(x - x^a) + \lambda L^\top L x = 0 \tag{21}$$

Assuming $W + \lambda L^\top L$ is nonsingular, the solution of (21) can be expressed by

$$\begin{aligned} x &= (G^\top G + \lambda L^\top L)^{-1} G^\top G x^a && (22) \\ &= (X\Sigma_G^2 X^\top + \lambda X \Sigma_L^2 X^\top)^{-1} X \Sigma_G^2 X^\top x^a \\ &= X^{-\top}(I + \lambda\Sigma)^{-1}\xi && (23) \end{aligned}$$

where $\Sigma = \Sigma_G^{-2}\Sigma_L^2$ and $\xi = X^\top x^a$. Substituting (23) into (18) yields

$$\begin{aligned} 0 &= \|Lx\|^2 - 1 \\ &= \left\|V\Sigma_L(I + \lambda\Sigma)^{-1}\xi\right\|^2 - 1 \\ &= \left\|\Sigma_L(I + \lambda\Sigma)^{-1}\xi\right\|^2 - 1 \ (V \text{ is orthonormal}) && (24) \\ &= \xi^\top (I + \lambda\Sigma)^{-\top}\Sigma_L^\top \Sigma_L (I + \lambda\Sigma)^{-1}\xi - 1 && (25) \end{aligned}$$

Notice that Σ_L and Σ_G are both diagonal, so is $I + \lambda\Sigma$. Therefore, Eq. (25) can be rewritten as

$$\xi^\top \Sigma_L^2 \rho(\lambda)\xi = 1 \tag{26}$$

where $\rho(\lambda) = (I + \lambda\Sigma)^{-2}$ is diagonal.

The above nonlinear equation (26) for the unknown λ can be solved numerically using Newton's method. First, $\rho(\lambda)$ can be approximated by the first two terms of its Taylor series expansion at $\lambda = \lambda_i$:

$$\rho(\lambda) \approx \rho(\lambda_i) + \dot{\rho}(\lambda_i) \cdot (\lambda - \lambda_i) \tag{27}$$

where $\dot{\rho}(\lambda_i)$ is the derivative of $\rho(\lambda)$ with respect to λ:

$$\dot{\rho}(\lambda) = -2\Sigma(I + \lambda\Sigma)^{-3} = -2\Sigma\rho(\lambda)(I + \lambda\Sigma)^{-1}$$

TABLE II
CALCULATING CONSTRAINED x BASED ON x^a

1.	Decompose Matrices: $W = G^\top G$, $G = U\Sigma_G X^\top$, $L = V\Sigma_L X^\top$
2.	Define: $\xi = X^\top x^a$ and $\Sigma = \Sigma_G^{-2}\Sigma_L^2$
3.	Calculate Multiplier λ: a) set the initial $\lambda_0 = 0$; b) while $\lvert\lambda_{i+1} - \lambda_i\rvert > \tau$, do iteration: $\rho(\lambda_i) = (I + \lambda_i\Sigma)^{-2}$, $\dot{\rho}(\lambda_i) = -2\Sigma\rho(\lambda_i)(I + \lambda_i\Sigma)^{-1}$, $\lambda_{i+1} = \lambda_i - \frac{\xi^\top\Sigma_L^2\rho(\lambda_i)\xi - 1}{2\xi^\top\Sigma_L^2\dot{\rho}(\lambda_i)\xi}$
4.	Obtain Constrained Optimum x: $x = X^{-\top}(I + \lambda_{i+1}\Sigma)^{-1}X^\top x^a$

because Σ and $I + \lambda\Sigma$ are diagonal. By substituting (27) into (26), the solution λ can be approximated by

$$\lambda_{i+1} = \lambda_i - \frac{\xi^\top\Sigma_L^2\rho(\lambda_i)\xi - 1}{2\xi^\top\Sigma_L^2\dot{\rho}(\lambda_i)\xi} \tag{28}$$

Next, expand $\rho(\lambda)$ at $\lambda = \lambda_{i+1}$ by the Taylor series expansion, and then repeat the above steps. The solution λ is obtained through the iteration until $|\lambda_{i+1} - \lambda_i| < \tau$, where τ is a threshold and λ_i is the λ value at iteration i.

In addition, it is well known that the initial condition of the iteration is critical for the convergence and the convergent rate of Newton's method. Before discussing the initial value of λ in (28), we first introduce the following proposition.

Proposition 1: In solving the constrained optimization problem (19), suppose that vectors $x^a_{(1)}$ and $x^a_{(2)}$ have the same optimization result x. If $x^a_{(1)}$ is closer to x than $x^a_{(2)}$ in the W-norm (i.e., $||x - x^a_{(1)}||_W < ||x - x^a_{(2)}||_W$), then multiplier $|\lambda_1| < |\lambda_2|$, where λ_j is associated with $x^a_{(j)}$. Particularly, if x^a coincides with the constrained state x, then $\lambda = 0$.

Proof: See Appendix A.

Proposition 1 shows that the multiplier λ continuously approaches zero as x^a converges to x for the QEC optimization problem. Since the given auxiliary state approximates the truth, it is suitable to set the initial $\lambda_0 = 0$. Finally, the constrained state x_k is obtained by substituting the above value of λ into Eq. (23), which corresponds to the optimization step labeled by "$\triangleright$" in Table I. For clarity, we summarize the solution procedure in Table II.

IV. ILLUSTRATIVE EXAMPLES AND DISCUSSIONS

In this section, using a road-based vehicle tracking example, we demonstrate the effectiveness of our proposed modeling approach and verify the theoretical results presented above.

In the simulation, a vehicle moves with a constant speed $s = 15\mathrm{m/s}$, and the speed constraint is

$$\dot{\mathsf{x}}^2 + \dot{\mathsf{y}}^2 = s^2 \tag{29}$$

where s is a known constant and $(\dot{\mathsf{x}}, \dot{\mathsf{y}})$ is the velocity. In addition, the velocity in the y-axis direction and the position are measured by

$$z_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + v_k \tag{30}$$

where the measurement noise $v_k \sim \mathcal{N}(0, R)$ is white and $R = \mathrm{diag}(400, 400, 20)$.

We model the constant speed motion by using the modeling framework (19). Assume that our auxiliary model is the typical constant velocity (CV) model

$$x^a_{k+1} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x^a_k + w_k \tag{31}$$

where the process noise $w_k \sim \mathcal{N}(0, Q^a)$ is white and $Q^a = \mathrm{diag}(16, 64, 1, 6)$. w_k and v_k are assumed to be mutually independent.

Rewrite the constraint (29) in the form of the normalized quadratic form (18) with $L = \frac{1}{s}\mathrm{diag}(0_2, I_2)$, set $W = I$, and then the solution to (21) is

$$x = \mathrm{diag}(I_2, (1 + \frac{\lambda}{s^2})^{-1} I_2) x^a \tag{32}$$

Putting (32) back into (29) yields $\lambda = s(s^a - s)$, where $s^a = \sqrt{(\dot{\mathsf{x}}^a)^2 + (\dot{\mathsf{y}}^a)^2}$. Therefore, the solution (32) is

$$x = \mathrm{diag}(I_2, \frac{s}{s^a} I_2) x^a \tag{33}$$

Finally, replacing the auxiliary x^a in (33) with the one in (31), we construct a dynamic model of the constant speed motion as follows

$$x_{k+1} = f(x_k, w_k) = \begin{bmatrix} \mathsf{x}_k + T\dot{\mathsf{x}}_k + w^1_k \\ \mathsf{y}_k + T\dot{\mathsf{y}}_k + w^2_k \\ (\dot{\mathsf{x}}_k + w^3_k)s/s^a_k \\ (\dot{\mathsf{y}}_k + w^4_k)s/s^a_k \end{bmatrix} \tag{34}$$

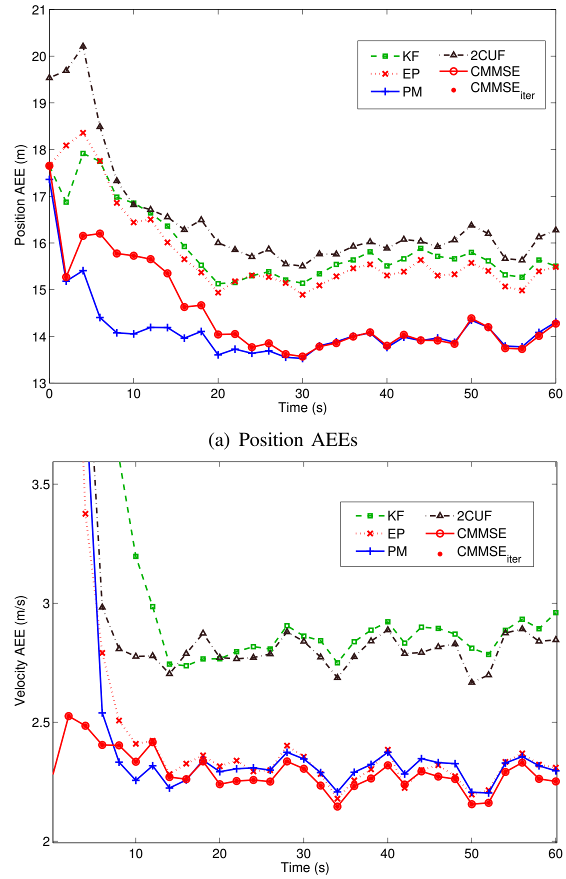
where $s^a_k = \sqrt{(\dot{\mathsf{x}}_k + w^3_k)^2 + (\dot{\mathsf{y}}_k + w^4_k)^2}$, and $w^{(i)}_k$ denotes the i-th component of the process noise vector w_k. The state in (34) satisfies the constraint (29). Set initial state $x_0 = [0, 0, 11.8301, 6.8301]^\top$.

Based on the constructed dynamic model along with the measurement model (30), we obtain the MMSE suboptimal estimate of the vehicle's state by using the unscented filter first, and then project the MMSE estimate onto the constraint surface. The performance of the proposed estimator labeled by "CMMSE" is shown by the red solid line with circles in Fig. 3. For this example, the Lagrangian multiplier λ can be solved analytically and the relationship between x and x^a is exactly determined by (33). Nevertheless, the iteration of λ presented in Section III-C is also performed. The performance of the CMMSE estimator based on the numerical solution is shown by the red stars and labeled by "CMMSE$_{\text{iter}}$" in the figure. Clearly, these two algorithms have basically the same results, which demonstrates the validity of our numerical solution of the quadratically constrained optimization.

Besides, four other conventional constrained estimators are also implemented, including the (unconstrained) Kalman filter, the estimate projection (EP) method, the pseudo measurement (PM) method, and two-step constrained unscented filter (2CUF) [1] [20]. They are all based on the same (unconstrained) system model (31)–(30) with the same initial state. In particular, differing from the EP that just projects the posterior estimate, the 2CUF projects the posterior sigma points onto the constraint surface, and then projects the weighted mean of the constrained sigma points to obtain the final constrained estimate. The average Euclidean error (AEE) of the state estimate is used to evaluate the estimation performance of different estimators [40]. Fig. 3 displays the comparison results of these estimation algorithms over 1000 Monte Carlo runs. From the figure, the following observations can be made:

1) Compared with the (unconstrained) KF, the five constrained estimators have lower estimation error levels, meaning that incorporation of the constraint information leads to better estimation performance.

2) The CMMSE and $\text{CMMSE}_{\text{iter}}$ have the lowest estimation error level in the steady state. In addition, the PM method performs closely to the CMMSE, but its estimate may violate the constraint.



(a) Position AEEs

(b) Velocity AEEs

Fig. 3. Performance assessment of the state estimation algorithms

V. Conclusions

Dynamic systems with equality constraints have been modeled in this paper. A minimum Euclidean distance criterion for sufficiently fusing the equality constraints and the auxiliary dynamics has been proposed, which is the core of the constrained dynamics modeling. Under this modeling framework, the constructed model of the dynamic systems with linear equality constraints has the same form as those using the state space decomposition technique, but our modeling is more intuitive and easier to implement. As a typical nonlinear equality constraint, we have focused on the quadratic equality constraint and presented the modeling for such a constrained dynamic system systematically. Additionally, it is found that the auxiliary dynamics affects the performance of the final constructed dynamic model to some extent. Although the preliminary analysis of the auxiliary dynamics has been presented, general principles for designing the auxiliary dynamics merit further study. In the scenario of tracking a road-based vehicle, our modeling method for the constrained motion has been employed, and by comparison with the conventional constrained estimation algorithms, the superiority of our constrained estimation algorithm based on the constructed model has been demonstrated in terms of accuracy.

Appendix

A. Proof of Proposition 1

As analyzed earlier, a first order necessary condition for minimizing (19) is given by (21). Since L is of full column rank, $L^\top L$ must be symmetric and positive definite. For any x, if x^a coincides with x then $\lambda = 0$.

Premultiplying $(x - x^a)^\top$ on both sides of Eq. (21) yields

$$\begin{aligned} &\|G(x-x^a)\|^2 + \lambda(x-x^a)^\top L^\top Lx \\ &= \|G(x-x^a)\|^2 + \lambda(1-(x^a)^\top L^\top Lx) = 0 \end{aligned} \tag{35}$$

If $x^a_{(1)}$ and $x^a_{(2)}$ satisfy $||x - x^a_{(1)}||^2_W < ||x - x^a_{(2)}||^2_W$, that is, $||G(x - x^a_{(1)})||^2 < ||G(x - x^a_{(2)})||^2$, from Eq. (35), it can be inferred that

$$\lambda_1(1-(x^a_1)^\top L^\top Lx) > \lambda_2(1-(x^a_2)^\top L^\top Lx) \tag{36}$$

Likewise, from Eq. (21), we have $x^a = x + \lambda(G^\top G)^{-1}L^\top Lx$. Then

$$\begin{aligned} (x^a)^\top L^\top Lx &= [x+\lambda(G^\top G)^{-1}L^\top Lx]^\top L^\top Lx \\ &= 1 + \lambda x^\top L^\top L(G^\top G)^{-1}L^\top Lx \end{aligned}$$

Substituting the above equation into inequality (36), we have

$$-\lambda_1^2 x^\top L^\top L(G^\top G)^{-1}L^\top Lx > -\lambda_2^2 x^\top L^\top L(G^\top G)^{-1}L^\top Lx$$

Since $L^\top L$ and $G^\top G$ are both positive definite, we can derive

$$\lambda_1^2 < \lambda_2^2$$

This completes the proof.

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