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Multiple-Model Estimation with Heterogeneous State Representation

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***Abstract*—How to fuse/combine state estimates that are obtained based on different models (e.g., a CV model, a CA model, and a CT model)? This paper provides a theoretical solution to such problems and beyond. Conventional multiple-model estimation methods use models defined in a common state space. In this paper, we discuss the advantage of using heterogeneous state space for different models in the multiple-model methods and deal with the consequent difficulties. Our algorithm is built mainly based on interacting multiple-model (IMM) due to its simplicity and popularity. Extensions to some other MM estimation methods, e.g., GPBn, are straightforward. For IMM with heterogeneous state, the model-conditioned estimates are converted to a common space for mixing and fusion. The reinitialization part is formulated as an optimization problem, which has an analytical solution. Our IMM with heterogeneous state is applied to a target tracking problem in a 2-dimensional scenario. Numerical results are provided to validate our method and demonstrate its performance compared with conventional IMM filters.**

***Index Terms*—Multiple-model estimation, IMM, target tracking, heterogeneous state space.**

I. Introduction

Multiple-model (MM) estimation is a powerful approach to adaptive estimation and is particularly good for problems involving structural as well as parametric changes [1]. It is a natural approach to estimate the state of a hybrid system, which has both continuous and discrete uncertainties in the base state and the mode, respectively [2]. MM estimation provides an integrated way to jointly handle both uncertainties and has the potential to achieve optimal performance. It has been studied intensively for decades, especially in target tacking [3], [4], [5], such as surveillance for air traffic control [6], [7] and maneuvering target tracking [8], [9].

The conventional MM method assumes a set of possible models and one of them is in effect at each time. It runs a bank of model-conditioned (or elemental) filters and generates the overall estimates based on the results of these filters [9]. Note that a model is a mathematical representation or description of a phenomenon or a system at a certain accurate level [9]. Thus, choosing states in different state spaces leads to different mathematical forms. These representations may or may not be equivalent. If we include models (e.g., constant velocity (CV), constant acceleration (CA) and constant turn (CT)) represented in different state spaces in the model set to implement an MM estimator, problems will arise in the state estimate combination (fusion) or mixing, which are the topic of this work. This differs significantly from the traditional MM estimation where all models are built in a common state space, that is, have a homogenous state representation. Here, we try to address the MM estimation with a heterogeneous state space. The advantage of using a heterogeneous state representation in MM for system identification has been discussed in [10], [11]. In this work, we exploit this more thoroughly for estimation.

It is well known that MM estimation algorithms have been classified into three generations [12], [9]. The first generation is autonomous MM (AMM) algorithms whose elemental filters operate individually and independently; the second generation is cooperating MM algorithms, represented by the Interacting MM (IMM), in which elemental filters cooperate effectively and work as a team; the third generation has a variable structure (VS), allows a variable set of models and is known as VSMM algorithms [9]. As pointed out in [1], using too many models is performance-wise as bad as using too few. So, the model set used in MM estimation should better be complete and compact. In other words, we need to include all possible models as well as the most accurate model in effect at a time in the model set and simultaneously make the size of the model-set as small as possible. In our opinion, using heterogeneous models can help achieve this better. Take target tracking as an example. Different target motion patterns may be better modeled in different state spaces, rather than in a common state space. For example, a turning motion may have a simpler form when modeled with state in the polar coordinate system (CS), while an acceleration motion may be more conveniently modeled in the Cartesian CS.

In this paper, we focus on the MM estimation based on systems with heterogeneous states. Model or model-set design to fully exploit the advantage of our algorithm is considered for our future work. We start from the assumption that a heterogeneous system is given and address the problem of state estimation in a specified space S. When the state $x \in S$ has linear relationships with the state in other spaces S^l, we show that the overall estimate of x can be optimal under the minimum mean-square error (MMSE) criterion in the AMM framework. When there is model switching, we need to handle the internal cooperation in the MM methods (e.g., reinitialization in the IMM algorithm), which is the major concern of this paper. For the nonlinear case, we adopt the natural idea of linearization.

As an application of our proposed method, a maneuvering target tracking problem with models described in different CS is considered. Although typical target motions are usually modeled in the Cartesian CS, such as the CV, CA and Singer models [13], there are still some models described in other CS (e.g., the decoupled first-order Markov models in the polar CS [14] and the decoupled model in the polar CS [15]). Since the true target motion is unknown, it is difficult to judge which one is more accurate. A practical way is to consider them all. Furthermore, using dynamic models in a sensor CS will lead to a simple (model-conditioned) filtering problem with linear uncoupled measurements in the Gaussian case, since measurements are available physically in a sensor CS [16]. In this paper, for simplicity we consider only the 2D case and use models in the Cartesian and the polar CS. We give necessary details in the Appendix for the reader to address 3D problems.

The paper is organized as follows. Section II gives a motivation example. Section III formulates the problem of MM estimation with heterogeneous state representation. Section IV presents the estimation method in the linear case and a simple extension to the nonlinear case. Our algorithm is applied to target tracking with models in different coordinate systems in Section V. Numerical results are provided in Section VI. Section VII concludes the paper. Some complemental mathematical details are given in the Appendix.

II. Motivation Example

We give an example here to motivate the consideration of heterogeneous models in MM estimation. Suppose a target has two possible motion patterns: CA and CT. So, in a 2D scenario, the following dynamic model is used for state estimation

$$x_{k+1} = F_k^{(l)} x_k + G_k^{(l)} w_k, \ l = 1, 2$$

where

$$\begin{aligned}
F_k^{(1)} &= \mathrm{diag}(\begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}) \\
F_k^{(2)} &= \begin{bmatrix} 1 & \frac{\sin\omega T}{\omega} & 0 & -\frac{1-\cos\omega T}{\omega} \\ 0 & \cos\omega T & 0 & -\sin\omega T \\ 0 & \frac{1-\cos\omega T}{\omega} & 1 & \frac{\sin\omega T}{\omega} \\ 0 & \sin\omega T & 0 & \cos\omega T \end{bmatrix} \\
G_k^{(1)} &= \mathrm{diag}(\begin{bmatrix} T^2/2 & T & 1 \end{bmatrix}', \begin{bmatrix} T^2/2 & T & 1 \end{bmatrix}') \\
G_k^{(2)} &= \mathrm{diag}(\begin{bmatrix} T^2/2 & T \end{bmatrix}', \begin{bmatrix} T^2/2 & T \end{bmatrix}')
\end{aligned}$$

T is the sampling interval and ω is the turn rate.

Note that the system states in the above models are different. The question is how to estimate the state of such a system by MM methods. This practically important problem, commonly encountered in maneuvering target tracking and unsolved yet, is considered in this paper. Instead of starting from this particular example, we formulate the system in a more general case and then give our solution.

III. Problem Formulation

Consider the following hybrid system

$$x_{k+1}^{(l)} = F_k^{(l)} x_k^{(l)} + G_k^{(l)} w_k^{(l)} \tag{1}$$

$$z_k = H_k^{(l)} x_k^{(l)} + v_k^{(l)}, \ l = 1, \dots, M \tag{2}$$

where $z_k \in \mathbb{R}^{n_z}$ and superscript (l) denotes quantities pertinent to model $m^{(l)}$. Different from the conventional Markovian jump linear system (MJLS), here the system states are related to system models. That is, different system models may be represented in different state spaces. Suppose we know the functions which relate all states $x_k^{(l)}, l = 1, \dots, M$, to a common specified state x_k:

$$x_k = \varphi_l(x_k^{(l)}) \tag{3}$$

The problem is how to estimate x_k given the measurements $z_{1:k} = [z_1', \dots, z_k']'$.

Note that x_k could be one of the $x^{(l)}$. Basically, x_k is in a state space of interest to the user. For example, $x_k^{(1)}$ and $x_k^{(2)}$ may be the state in the Cartesian and the polar CS, respectively. If we are interested in estimation in the Cartesian CS, we may choose $x_k = x_k^{(1)}$. Common practice in this case would build all the models in the Cartesian states (e.g., Cartesian position and velocity). However, this may not necessarily lead to good performance. As seen in [13], turning models may have simpler forms with the polar state, and acceleration models are usually better represented in the Cartesian state. Although models can be converted based on state conversion, this may lead to a complicated model or degraded accuracy, and hence may make the estimator suffer.

We further have the following assumption of the above system. The event that model $m^{(l)}$ is in effect over the sampling interval $(t_{k-1}, t_k]$ will be denoted by $m_k^{(l)}$. The model sequence is assumed to be a homogeneous finite-state Markovian chain with transition probabilities $\pi_{ij} = P(m_k^{(j)} | m_{k-1}^{(i)})$. The transition probability matrix $[\pi_{ij}]$ is an $M \times M$ matrix with all elements satisfying $0 \le \pi_{ij} \le 1$ and $\sum_j \pi_{ij} = 1$, where M is the number of system models. The process noise $w^{(l)}$ and the measurement noise $v^{(l)}$ are mutually independent zero-mean white Gaussian processes with covariance matrices

$$\mathrm{cov}(w_k^{(l)}) = Q_k^{(l)}, \ \mathrm{cov}(v_k^{(l)}) = R_k^{(l)}$$

The initial state x_0 is assumed to be independent of $w^{(l)}$ and $v^{(l)}$.

Note that when $x_k^{(l)} = x_k$ for all l, the system (1)-(2) reduces to a traditional MJLS and there are abundant MM methods under different criteria (e.g., MMSE and maximum a posteriori) to estimate the state of such a system. Many suboptimal methods under the MMSE criterion are available, such as GPBn and IMM. Among these, IMM is most popular since it usually achieves a good accuracy with relatively small computational complexity.

As to our problem, we still use the IMM strategy and address the problems encountered in its implementation. We give a brief cycle of IMM estimator here for the convenience of discussion later and the reader is referred to [9] for more details. A standard IMM estimator for system (1)-(2) with $x_k^{(l)} = x_k$ consists of the following four steps:

1) *Model-conditioned reinitialization.*

Compute the predicted model probability

$$\mu_{k|k-1}^{(l)} \triangleq P\{m_k^{(l)}, z^{k-1}\}$$

mixing weight

$$\mu_{k-1}^{j|l} \triangleq P\{m_{k-1}^{(j)} | m_k^{(l)}, z_{k-1}\}$$

mixed estimate

$$\bar{x}_{k-1|k-1}^{(l)} \triangleq E[x_{k-1} | m_k^{(l)}, z^{k-1}]$$

and its corresponding MSE matrix.

2) *Model-conditioned filtering.*

For each model $m_k^{(l)}$, do model-conditioned estimation:

$$\hat{x}_{k|k}^{(l)} = E[x_k | m_k^{(l)}, z^k]$$

3) *Model probability update.*

Compute the probability $\mu_k^{(l)}$ of the event $m_k^{(l)}$.

4) *Output the overall estimate.*

Compute the overall estimate

$$\hat{x}_{k|k} = \sum_l \mu_k^{(l)} \hat{x}_{k|k}^{(l)}$$

This standard IMM estimator is based on homogenous models, that is, all models are represented in space S. For our formulation, we use the following NOMENCLATURE:

$\hat{x}_{k\|k}^{(l)}$	Model $m_k^{(l)}$ conditioned estimate of x_k in S
$\check{x}_{k\|k}^{(l)}$	Model $m_k^{(l)}$ conditioned estimate of $x_k^{(l)}$ in S^l
$\bar{x}_{k\|k}^{(l)}$	Mixed estimate of x_k in S
$\acute{x}_{k\|k}^{(l)}$	Mixed estimate of $x_k^{(l)}$ in S^l

These definitions will be repeated when necessary to reduce possible confusion.

IV. IMM Estimation for Heterogeneous States

A. Linear Case

Here, we consider linear functions between model-conditioned state $x_k^{(l)}$, $l = 1, \ldots, M$, and the common state x_k (in the state space of interest to us)

$$x_k = \Phi_l x_k^{(l)} \tag{4}$$

Since expectation is a linear operator, we have the model-conditioned estimation as

$$\begin{aligned} \hat{x}_k^{(l)} &= E[x_k | m_k^{(l)}, z^k] \\ &= \Phi_l E[x_k^{(l)} | m_k^{(l)}, z^k] = \Phi_l \check{x}_k^{(l)} \end{aligned} \tag{5}$$

So, from (5), the (model-conditioned) estimate $\hat{x}_k^{(l)}$ of x_k can be obtained by just transforming the (model-conditioned) estimate $\check{x}_k^{(l)}$ of $x_k^{(l)}$ through Φ_l.

If there is no model switching (i.e., under the AMM assumption), the optimality of $\hat{x}_{k|k}$ in the MMSE sense can be guaranteed. If there is model switching, as we assumed in Section III, the standard IMM procedure can be followed except for the difficulty of reinitializing the model-based filter.

We use $\check{x}_{k|k}^{(l)}$ to denote the model $m_k^{(l)}$ conditioned estimate of $x_k^{(l)}$ in space S^l. Suppose we know

$$\mu_k^{j|i} \triangleq P\{m_k^{(j)} | m_{k+1}^{(i)}, z^k\}$$

and the mixed estimate of x_k can be obtained as

$$\begin{aligned} \bar{x}_{k|k}^{(i)} &= E[x_k | m_{k+1}^{(i)}, z^k] \\ &= \sum_j \mu_k^{j|i} E[x_k | m_{k+1}^{(i)}, m_k^{(j)}, z^k] = \sum_j \mu_k^{j|i} \hat{x}_{k|k}^{(j)} \\ &= \sum_j \mu_k^{j|i} \Phi_j E[x_k^{(j)} | m_{k+1}^{(i)}, m_k^{(j)}, z^k] = \sum_j \mu_k^{j|i} \Phi_j \check{x}_{k|k}^{(j)} \end{aligned}$$

where $\hat{x}_{k|k}^{(j)}$ denotes the model $m_k^{(j)}$ conditioned estimate of $x_k \in S$. Given this mixed estimate $\bar{x}_{k|k}^{(i)}$, we want to find its "equivalent" (denoted as $\acute{x}_{k|k}^{(i)}$) in space S^i to reinitialize the elemental filters. To do this, the only available relationship is (4) (i.e., $\bar{x}_{k|k}^{(i)} = \Phi_i \acute{x}_{k|k}^{(i)}$). Here, we do not want to impose any limitations upon Φ_l and thus the solution of $\acute{x}_{k|k}^{(l)}$ through (4) may not be existent or unique. We propose the following way to get an unique $\acute{x}_{k|k}^{(l)}$ when a solution exists:

$$\min_{\acute{x}_{k|k}^{(l)}} \left\| \acute{x}_{k|k}^{(l)} - \check{x}_{k|k}^{(l)} \right\| \tag{6}$$

$$\text{s.t. } \Phi_l \acute{x}_{k|k}^{(l)} = \bar{x}_{k|k}^{(l)} \tag{7}$$

where $\|\cdot\|$ denotes the Euclidean norm. When the constraint (7) cannot be satisfied, see *Remark 2* below for details.

An analytic solution to the above problem is (see the Appendix A for a derivation)

$$\acute{x}_{k|k}^{(l)} = \Phi_l^+ \bar{x}_{k|k}^{(l)} + (I - \Phi_l^+ \Phi_l) \check{x}_{k|k}^{(l)} \tag{8}$$

where the superscript "+" stands for the Moore-Penrose pseudo-inverse (MP inverse for short) and I is an identity matrix with a compatible dimension. Once this mixed estimate $\acute{x}_{k|k}^{(l)}$ in S^l is obtained, the model-conditioned prediction can be carried out as

$$\check{x}_{k+1|k}^{(l)} = F_k^{(l)} \acute{x}_{k|k}^{(l)}$$

So far, for the problem considered, we propose to implement the mixing and fusion parts in space S and reinitialize the model-conditioned filter by (8). In the following, we address the MSE matrices of the above estimates. Given the following MSE matrices

$$\check{P}_{k|k}^{(l)} = \text{MSE}(\check{x}_{k|k}^{(l)}), \ l = 1, \ldots, M$$

we have

$$\bar{P}_{k|k}^{(i)} \triangleq \text{MSE}(\bar{x}_{k|k}^{(i)}) = \sum_j \mu_k^{j|i}[\Phi_j \check{P}_{k|k}^{(j)} \Phi_j' + \tilde{x}_{k|k}^{(i)}(\tilde{x}_{k|k}^{(i)})'] \quad (9)$$

where

$$\tilde{x}_{k|k}^{(i)} = \bar{x}_{k|k}^{(i)} - \Phi_j \check{x}_{k|k}^{(j)}$$

Since

$$\begin{aligned}
&\acute{x}_{k|k}^{(l)} - x_k^{(l)} \\
&= \Phi_l^+ \bar{x}_{k|k}^{(l)} + (I - \Phi_l^+ \Phi_l)\check{x}_{k|k}^{(l)} - x_k^{(l)} \\
&= \Phi_l^+ (\bar{x}_{k|k}^{(l)} - x_k) + (I - \Phi_l^+ \Phi_l)\check{x}_{k|k}^{(l)} - x_k^{(l)} + \Phi_l^+ x_k \\
&= \Phi_l^+ (\bar{x}_{k|k}^{(l)} - x_k) + (I - \Phi_l^+ \Phi_l)(\check{x}_{k|k}^{(l)} - x_k^{(l)})
\end{aligned}$$

we ignore the cross-covariance between $\bar{x}_{k|k}^{(l)} - x_k$ and $\check{x}_{k|k}^{(l)} - x_k^{(l)}$, and thus the MSE matrix $\acute{P}_{k|k}^{(l)}$ of $\acute{x}_{k|k}^{(l)}$ is

$$\acute{P}_{k|k}^{(l)} \approx \Phi_l^+ \bar{P}_{k|k}^{(l)} (\Phi_l^+)' + (I - \Phi_l^+ \Phi_l)\check{P}_{k|k}^{(l)} (I - \Phi_l^+ \Phi_l)' \quad (10)$$

Remark 1: When all Φ_l, $l = 1, \dots, M$, are invertible, the problem is easy to handle. We can transform all dynamic models to space S and then use the standard IMM algorithm. Our solution reduces to this special case, since (8) becomes

$$\begin{aligned}
\acute{x}_{k|k}^{(l)} &= \Phi_l^{-1} \bar{x}_{k|k}^{(l)} + (I - \Phi_l^{-1}\Phi_l)\check{x}_{k|k}^{(l)} \\
&= \Phi_l^{-1} \bar{x}_{k|k}^{(l)} \qquad (11)
\end{aligned}$$

Remark 2: When $\Phi_l \acute{x}_{k|k}^{(l)} = \bar{x}_{k|k}^{(l)}$ is an inconsistent system of equations, that is, $\bar{x}_{k|k}^{(l)}$ is not in the range of Φ_l, there exists no solution $\acute{x}_{k|k}^{(l)}$ of (6)-(7). However, there is an approximate solution which minimizes the squared error of the difference between the two sides of equation (7). That is, the optimization problem (6)-(7) is replaced by

$$\min_{\acute{x}_{k|k}^{(l)}} \left\| \acute{x}_{k|k}^{(l)} - \check{x}_{k|k}^{(l)} \right\| \quad (12)$$
$$\text{s.t.} \quad \left\| \Phi_l \acute{x}_{k|k}^{(l)} - \bar{x}_{k|k}^{(l)} \right\| = \inf_x \left\| \Phi_l x - \bar{x}_{k|k}^{(l)} \right\|$$

The solution is still given by (8) by the minimum norm least squares (see Appendix B for details). This reveals an advantage of using Moore-Penrose inverse — the solutions in both consistent and inconsistent cases are unified.

Remark 3: Suppose S^l is a subspace of S and $\begin{bmatrix} (x^{(l)})' & y' \end{bmatrix}' \in S$, where $y \in \bar{S}^l$, $\bar{S}^l \otimes S^l = S$ and $\otimes$ denotes the direct product. For this case, we may set

$$\Phi_l = \begin{bmatrix} I & \mathbf{0} \end{bmatrix}' \quad (13)$$

It can be shown that our algorithm based on Φ_l is equivalent to applying IMM to the following dynamic model (details are omitted here for lack of space):

$$x_{k+1} = \bar{F}_k^{(l)} x_k + \bar{G}_k^{(l)} w_k^{(l)} \quad (14)$$
$$z_k = \bar{H}_k^{(l)} x_k + v_k^{(l)} \quad (15)$$

where

$$\bar{F}_k^{(l)} = \begin{bmatrix} F_k^{(l)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \bar{G}_k^{(l)} = \begin{bmatrix} G_k^{(l)} \\ \mathbf{0} \end{bmatrix}$$
$$\bar{H}_k^{(l)} = \begin{bmatrix} H_k^{(l)} & \mathbf{0} \end{bmatrix}$$

This happens, for example, when we use CV and CA models in an MM estimation.

Note, however, that it is not always right to set Φ_l as (13) in the case where S^l is a subspace of S. For the motivation example in Section II, we use

$$x^{(1)} = [\mathsf{x}, \dot{\mathsf{x}}, \ddot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}, \ddot{\mathsf{y}}]' \in S^1$$
$$x^{(2)} = [\mathtt{x}, \dot{\mathtt{x}}, \mathtt{y}, \dot{\mathtt{y}}]' \in S^2$$

to denote the state of CA model and the state of CT model, respectively. Choose $S = S^1$ and then we have the following relationship between $x^{(1)}$ and $x^{(2)}$:

$$\begin{aligned}
&\mathsf{x} = \mathtt{x}, \ \dot{\mathsf{x}} = \dot{\mathtt{x}} \\
&\mathsf{y} = \mathtt{y}, \ \dot{\mathsf{y}} = \dot{\mathtt{y}} \\
&\ddot{\mathsf{x}} = \omega\sqrt{\dot{\mathtt{x}}^2 + \dot{\mathtt{y}}^2}\cos(\arctan(\frac{\dot{\mathtt{y}}}{\dot{\mathtt{x}}}) + \frac{\pi}{2}\text{sgn}(\omega)) \\
&\ddot{\mathsf{y}} = \omega\sqrt{\dot{\mathtt{x}}^2 + \dot{\mathtt{y}}^2}\sin(\arctan(\frac{\dot{\mathtt{y}}}{\dot{\mathtt{x}}}) + \frac{\pi}{2}\text{sgn}(\omega))
\end{aligned}$$

Although S^2 is a subspace of S^1, their states have the above nonlinear relationship, which is addressed next.

B. Nonlinear Case

For the nonlinear case of (3), a practical way is to linearize. The mixed estimate of x_k and its corresponding MSE can be calculated as

$$\bar{x}_{k|k}^{(i)} = E[x_k | m_{k+1}^{(i)}, z^k] \approx \sum_j \mu_k^{j|i} \varphi_j(\check{x}_{k|k}^{(j)}) \quad (16)$$

$$\bar{P}_{k|k}^{(i)} \triangleq \text{MSE}(\bar{x}_{k|k}^{(i)}) \approx \sum_j \mu_k^{j|i}[\Psi_j \check{P}_{k|k}^{(j)} \Psi_j' + \tilde{x}_{k|k}^{(i)}(\tilde{x}_{k|k}^{(i)})'] \quad (17)$$

where

$$\tilde{x}_{k|k}^{(i)} = \bar{x}_{k|k}^{(i)} - \varphi_j(\check{x}_{k|k}^{(j)})$$

and $\Psi_j = \partial\varphi_j(x)/\partial x|_{x=\check{x}_{k|k}^{(j)}}$.

For the mixed estimate $\acute{x}_{k|k}^{(l)}$ in S^i, the solution depends on the existence of the inverse function $x_k^{(l)} = \varphi_l^{-1}(x_k)$.

1) $\varphi_l^{-1}(\cdot)$ is available: We have

$$\acute{x}_{k|k}^{(l)} \approx \varphi_l^{-}(\bar{x}_{k|k}^{(l)}) \quad (18)$$
$$\acute{P}_{k|k}^{(l)} \approx \Psi_l^{-} \bar{P}_{k|k}^{(l)} (\Psi_l^{-})' \quad (19)$$

where $\Psi_l^{-} = \partial\varphi_l^{-1}(x)/\partial x|_{x=\bar{x}_{k|k}^{(l)}}$.

Note that for Eqs. (16)–(19), unscented transform (UT) [17] is a better method to approximate the means and MSE matrices at the cost of higher computation complexity. However, UT is not applicable when $\varphi_l^{-1}(\cdot)$ is not available.

2) $\varphi_l^{-1}(\cdot)$ is not available: We first linearize (3) at $\check{x}_{k|k}^{(l)}$ and then use our result for the linear case to get the following solution:

$$\begin{aligned}\acute{x}_{k|k}^{(l)} &\approx \Psi_l^{+}[\bar{x}_{k|k}^{(l)} - \varphi_l(\check{x}_{k|k}^{(l)})] + \check{x}_{k|k}^{(l)} \\ \acute{P}_{k|k}^{(l)} &\approx \Psi_l^{+}\bar{P}_{k|k}^{(l)}(\Psi_l^{+})' + (I - \Psi_l^{+}\Psi_l)\check{P}_{k|k}^{(l)}(I - \Psi_l^{+}\Psi_l)'\end{aligned}$$

V. Target Tracking with Models in Different Coordinate Systems

In this section, we apply our method to maneuvering target tracking with models in different CS. For simplicity of discussion, we only consider a 2D scenario. Solutions for the 3D case can be directly obtained by following the same procedure. The necessary details are provided in Appendices C and D.

We assume the two models used in an IMM filter for tracking are established in the Cartesian CS and the polar CS, respectively. That is,

$$x_k^{(i)} = \begin{bmatrix} \mathsf{x}_k \\ \dot{\mathsf{x}}_k \\ \mathsf{y}_k \\ \dot{\mathsf{y}}_k \end{bmatrix}, \; x_k^{(j)} = \begin{bmatrix} r_k \\ \dot{r}_k \\ b_k \\ \dot{b}_k \end{bmatrix}$$

where $(\mathsf{x}_k, \mathsf{y}_k)$ is the Cartesian position, $(\dot{\mathsf{x}}_k, \dot{\mathsf{y}}_k)$ the Cartesian velocity, (r_k, b_k) the polar position (range, bearing), and $(\dot{r}_k, \dot{b}_k)$ the polar velocity (range rate, bearing rate). Both CS have the common original point and b is the angle between range direction and the x axis. The reader is referred to Fig.1 in [16] for an illustration in the 3D case. Suppose the reference state is $x_k = x_k^{(j)}$ and then x_k can be obtained by the following nonlinear one-to-one transformation:

$$\begin{aligned}x_k = \varphi_i(x_k^{(i)}) &= \begin{bmatrix} \sqrt{\mathsf{x}_k^2 + \mathsf{y}_k^2} \\ \dot{\mathsf{x}}_k \cos b_k + \dot{\mathsf{y}}_k \sin b_k \\ \arctan(\frac{\mathsf{y}_k}{\mathsf{x}_k}) \\ \frac{\dot{\mathsf{y}}_k \mathsf{x}_k - \mathsf{y}_k \dot{\mathsf{x}}_k}{\mathsf{x}_k^2 + \mathsf{y}_k^2} \end{bmatrix} \\ x_k = \varphi_j(x_k^{(j)}) &= I x_k^{(j)}\end{aligned}$$

So, given the model-conditioned estimates $\check{x}_{k|k}^{(l)}$, $l = i, j$, at time $k-1$ and their corresponding MSE matrices $\check{P}_{k|k}^{(l)}$, by (16) we have the mixed estimate in the polar CS as

$$\bar{x}_{k|k}^{(l)} \approx \mu_k^{j|l}\check{x}_{k|k}^{(j)} + \mu_k^{i|l}\varphi_i(\check{x}_{k|k}^{(i)})$$

and by (17)

$$\begin{aligned}\bar{P}_{k|k}^{(l)} \approx\; & \mu_k^{j|l}[\check{P}_{k|k}^{(j)} + \tilde{x}_{k|k}^{(j)}(\tilde{x}_{k|k}^{(j)})'] \\ & + \mu_k^{i|l}[\Psi_i \check{P}_{k|k}^{(i)}\Psi_i' + \tilde{x}_{k|k}^{(i)}(\tilde{x}_{k|k}^{(i)})']\end{aligned}$$

where

$$\begin{aligned}\tilde{x}_{k|k}^{(i)} &= \bar{x}_{k|k}^{(l)} - \varphi_i(\check{x}_{k|k}^{(i)}) \\ \tilde{x}_{k|k}^{(j)} &= \bar{x}_{k|k}^{(l)} - \check{x}_{k|k}^{(j)}\end{aligned}$$

and the calculation of matrix Ψ_i is given next.

Suppose $\check{x}_{k|k}^{(i)} = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}]'$. Then

$$\begin{aligned}\Psi_i &= \frac{\partial \varphi_i(x)}{\partial x}\Big|_{x=\check{x}_{k|k}^{(i)}} \\ &= \begin{bmatrix} \frac{\mathsf{x}}{\sqrt{\mathsf{x}^2+\mathsf{y}^2}} & 0 & \frac{\mathsf{y}}{\sqrt{\mathsf{x}^2+\mathsf{y}^2}} & 0 \\ U & \cos b & V & \sin b \\ \frac{-\mathsf{y}}{\mathsf{x}^2+\mathsf{y}^2} & 0 & \frac{\mathsf{x}}{\mathsf{x}^2+\mathsf{y}^2} & 0 \\ W & \frac{-\mathsf{y}}{\mathsf{x}^2+\mathsf{y}^2} & T & \frac{\mathsf{x}}{\mathsf{x}^2+\mathsf{y}^2} \end{bmatrix}\end{aligned}$$

where

$$\begin{aligned}b &= \arctan(\frac{\mathsf{y}}{\mathsf{x}}) \\ U &= \frac{\mathsf{y}\dot{\mathsf{x}}\sin b}{\mathsf{x}^2+\mathsf{y}^2} - \frac{\mathsf{y}\dot{\mathsf{y}}\cos b}{\mathsf{x}^2+\mathsf{y}^2} \\ V &= \frac{-\mathsf{x}\dot{\mathsf{x}}\sin b}{\mathsf{x}^2+\mathsf{y}^2} + \frac{\mathsf{x}\dot{\mathsf{y}}\cos b}{\mathsf{x}^2+\mathsf{y}^2} \\ W &= \frac{\dot{\mathsf{y}}}{\mathsf{x}^2+\mathsf{y}^2} - \frac{2\mathsf{x}(\dot{\mathsf{y}}\mathsf{x} - \mathsf{y}\dot{\mathsf{x}})}{(\mathsf{x}^2+\mathsf{y}^2)^2} \\ T &= \frac{-\dot{\mathsf{x}}}{\mathsf{x}^2+\mathsf{y}^2} - \frac{2\mathsf{y}(\dot{\mathsf{y}}\mathsf{x} - \mathsf{y}\dot{\mathsf{x}})}{(\mathsf{x}^2+\mathsf{y}^2)^2}\end{aligned}$$

Now, we need to calculate the mixed estimates $(\acute{x}_{k|k}^{(l)}, \acute{P}_{k|k}^{(l)}), l = i, j$, in S^l for the reinitialization of the model-conditioned filters. Note that $\varphi_l^{-1}(\cdot), l = i, j$, are known as

$$\begin{aligned}x_k^{(i)} = \varphi_i^{-1}(x_k) &= \begin{bmatrix} r_k \cos b_k \\ \dot{r}_k \cos b_k - \dot{b}_k r_k \sin b_k \\ r_k \sin b_k \\ \dot{r}_k \sin b_k + \dot{b}_k r_k \cos b_k \end{bmatrix} \\ x_k^{(j)} = \varphi_j^{-1}(x_k) &= I x_k\end{aligned}$$

Thus, by (18)-(19) we have

$$\begin{aligned}\acute{x}_{k|k}^{(i)} &\approx \varphi_i^{-1}(\bar{x}_{k|k}^{(i)}) \\ \acute{P}_{k|k}^{(i)} &\approx \Psi_i^{-}\bar{P}_{k|k}^{(i)}(\Psi_i^{-})' \\ \acute{x}_{k|k}^{(j)} &= \bar{x}_{k|k}^{(j)} \\ \acute{P}_{k|k}^{(j)} &= \bar{P}_{k|k}^{(j)}\end{aligned}$$

where the matrix Ψ_i^{-} is given next. Suppose $\bar{x}_{k|k}^{(i)} = \left[r, \dot{r}, b, \dot{b}\right]'$. Then, we have

$$\begin{aligned}\Psi_i^{-} &= \frac{\partial \varphi_i^{-1}(x)}{\partial x}\Big|_{x=\bar{x}_{k|k}^{(i)}} \\ &= \begin{bmatrix} \cos b & 0 & -r\sin b & 0 \\ -\dot{b}\sin b & \cos b & -\dot{r}\sin b - \dot{b}r\cos b & -r\sin b \\ \sin b & 0 & r\cos b & 0 \\ \dot{b}\cos b & \sin b & \dot{r}\cos b + \dot{b}r\sin b & r\cos b \end{bmatrix}\end{aligned}$$

So far we have addressed the problem of target tracking using models in the 2D Cartesian and polar CS. For 3D and RUV measurements, the functions $\varphi_l(\cdot)$ and $\varphi_l^{-1}(\cdot)$ are given in Appendices C and D.

VI. Numerical Examples

An illustrative example of maneuvering target tracking is provided to validate our proposed multiple-model algorithm. Since average Euclidean error (AEE) is more advantageous than root mean square error (RMSE), as discussed in [18], we use AEE to evaluate performance. The results were obtained based on 500 runs of Monte Carlo simulation.

A. True Trajectory

Two models were used to generate the true trajectory over a total of 70s. In the first 20s, the target started from x_0 with a CV motion (in the Cartesian CS) as follows:

$$\begin{aligned} x_0 &\sim \mathcal{N}([90\text{m}, 90\text{m/s}, 50\text{m}, 50\text{m/s}]', \text{diag}(100, 1, 100, 1)) \\ x_{k+1} &= F_k^{(1)} x_k + G_k^{(1)} w_k^{(1)} \\ z_k &= \begin{bmatrix} \sqrt{(\mathsf{x}_k)^2 + (\mathsf{y}_k)^2} \\ \arctan(\mathsf{x}_k/\mathsf{y}_k) \end{bmatrix} + v_k \end{aligned} \tag{20}$$

where

$$F_k^{(1)} = \text{diag}(\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix})$$

$$G_k^{(1)} = \text{diag}(\begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}, \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix})$$

and T is the sampling interval.

From 21s to 45s, the target performed a motion (see, e.g., [15]) in the polar CS:

$$\begin{aligned} x_{k+1} &= F_k^{(2)} x_k + G_k^{(2)} [w_k^{(2)} + u_k] \\ z_k &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_k + v_k \end{aligned} \tag{21}$$

where

$$F_k^{(2)} = \text{diag}\{\begin{bmatrix} 1 & \rho_1 \\ 0 & \acute{\rho}_1 \end{bmatrix}, \begin{bmatrix} 1 & \rho_1 \\ 0 & \acute{\rho}_1 \end{bmatrix}\}$$

$$G_k^{(1)} = \text{diag}\{\begin{bmatrix} \rho_2 \\ \acute{\rho}_2 \end{bmatrix}, \begin{bmatrix} \rho_2/x_k^1 \\ \acute{\rho}_2/x_k^1 \end{bmatrix}\}$$

and

$$\rho_1 = \frac{(1 - e^{-\alpha T})}{\alpha}, \; \acute{\rho}_1 = e^{\alpha T}$$

$$\rho_2 = \frac{(e^{-\alpha T} - 1 + \alpha T)}{2\alpha}, \; \acute{\rho}_2 = \frac{(1 - e^{-\alpha T})}{2}$$

In the remaining 25s, the target motion switched to model (20).

In the simulation, we set $T = 1\text{s}$, $\alpha = 0.03$, $u_k = [100, 160]'$ and

$$\begin{aligned} w_k^{(1)} &\sim \mathcal{N}([0, 0]', \text{diag}(0.1^2 (\text{m/s})^2, 0.1^2 (\text{m/s})^2)) \\ w_k^{(2)} &\sim \mathcal{N}([0, 0]', \text{diag}(0.1^2 (\text{m/s})^2, 0.001^2 (\text{rad/s})^2)) \\ v_k &\sim \mathcal{N}([0, 0]', \text{diag}(6^2 \text{m}^2, 0.006^2 (\text{rad/s})^2)) \end{aligned}$$

In the three time intervals, the true states were all converted to the polar CS for performance evaluation.

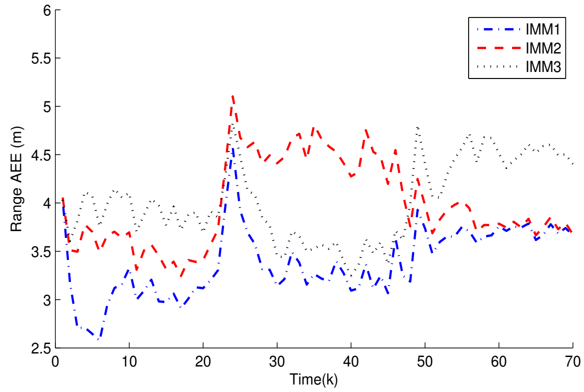

Fig. 1. Range AEEs.

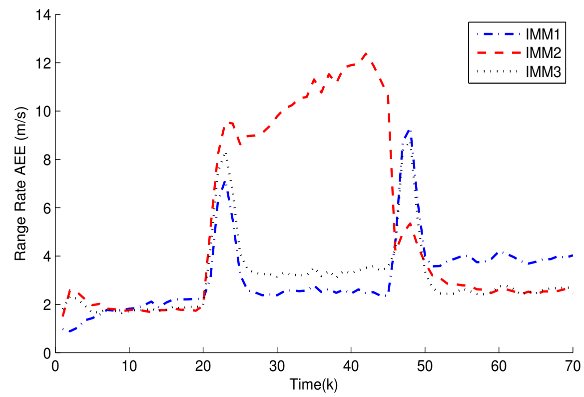

Fig. 2. Range Rate AEEs.

B. Filter Design

Three IMM filters with different model sets were used to estimate the target state. The first IMM filter (denoted by IMM1) uses the two true models (20) and (21) with heterogeneous representation. The second IMM filter (denoted by IMM2) uses five homogeneous Cartesian models: four CV models with different levels of Q, and CT with an unknown turn rate. The third IMM filter (denoted by IMM3) uses four heterogeneous models: model (20), CV with Q_{CV4}, CT with a known turn rate, and model (21).

In the simulation, the extended Kalman filter (EKF) was used to handle all nonlinear model-conditioned filtering.

The covariances of the zero-mean process noise in the CV model were set as

$$Q_{\text{CV1}} = \text{diag}(0.1^2, 0.1^2) \tag{22}$$

$$Q_{\text{CV2}} = \text{diag}(4^2, 4^2) \tag{23}$$

$$Q_{\text{CV3}} = \text{diag}(8^2, 8^2) \tag{24}$$

$$Q_{\text{CV4}} = \text{diag}(12^2, 12^2) \tag{25}$$

and for the CT model with known turn rate, the turn rate and

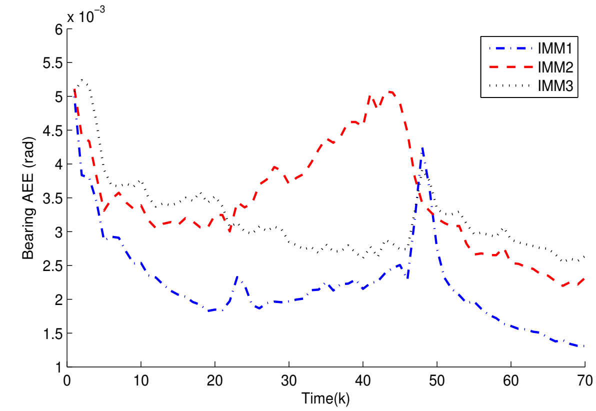

Fig. 3. Bearing AEEs.

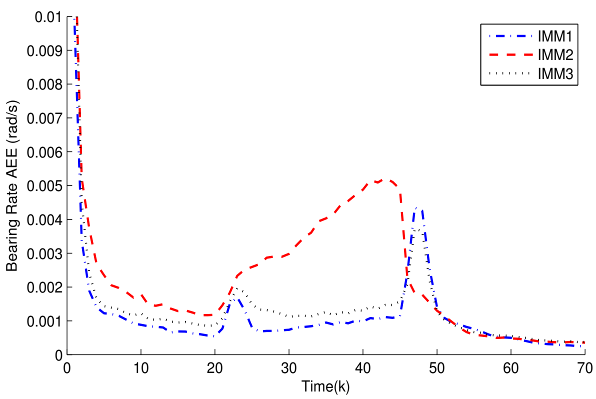

Fig. 4. Bearing Rate AEEs.

the covariance of the zero-mean process noise were set as

$$\omega = 0.03\text{rad/s}$$
$$Q_{\text{CT}} = \text{diag}(2^2, 2^2)$$

For the elemental filtering based on the CT model with an unknown turn rate, an adaptive method was adopted (see, e.g., [19], [13] for details). To be fair for all three MM filters, we have tuned their covariances and ω to achieve the best performance. The transition probability matrices of the three filters were set as

$$\Pi_{\text{IMM1}} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$
$$\Pi_{\text{IMM2}} = [\pi_{ij}]_{5\times5} = \begin{cases} 0.8, i = j \\ 0.05, i \neq j \end{cases}$$
$$\Pi_{\text{IMM3}} = [\pi_{ij}]_{4\times4} = \begin{cases} 0.85, i = j \\ 0.05, i \neq j \end{cases}$$

C. Simulation Result

The range, range rate, bearing, and bearing rate AEEs of the three IMM filters are shown in Fig. 1, Fig. 2, Fig. 3 and Fig. 4, respectively.

From these figures, we can see that IMM2 is generally worse than IMM1, except that it slightly outperforms IMM1 on range rate estimation in steady state during CV motion. This indicates the superiority of using heterogeneous models in MM methods. For IMM1 and IMM3, they both contain the true heterogeneous models. However, since IMM3 used two mismatched models, IMM1 outperforms IMM3 as shown in these figures. This should be the case in theory [1], as stated in the Introduction.

VII. Conclusions

In this paper, MM estimation with models in heterogeneous state spaces has been studied. We extended the IMM algorithm to deal with the problem caused by heterogeneous state. If linear mappings exist between the reference state and the model-conditioned states, our solutions for mixing and fusion in IMM algorithm are analytical. For nonlinear mappings, linearization based on Taylor series expansion or unscented transformation can be applied.

A model is a highly condensed form of much useful prior information. If it is accurate, we should try to use it as what it is. In our proposed method, we do not convert models but estimates. So, we actually take good advantage of model information by using heterogeneous models rather than converting all models into a common state space.

Appendix

A. Solution to (6)

We drop the unnecessary subscripts and superscripts for clarity and write the problem (6) as

$$\min_x ||x - b|| \quad (26)$$
$$\text{s.t. } \Phi x = a$$

Let $y \triangleq x - b$. The original optimization problem becomes

$$\min_y ||y|| \quad (27)$$
$$\text{s.t. } \Phi y = a - \Phi b$$

which amounts to finding the minimum norm solution of the consistent equation $\Phi y = a - \Phi b$.

For this problem, $G(a - \Phi b)$ is the solution of y if and only if [20]

$$\Phi G \Phi = \Phi, \ (G\Phi)' = G\Phi$$

Note that G is not unique but the solution is. Clearly, $G = \Phi^+$ satisfies the above requirements and thus we have the solution

$$x = \Phi^+ a + (I - \Phi^+ \Phi)b$$

B. Solution to (12)

We drop the unnecessary subscripts and superscripts for clarity and write the problem (6) as

$$\min_x ||x - b|| \quad (28)$$
$$\text{s.t. } \|\Phi x - a\| = \inf_v \|\Phi v - a\|$$

Let $y \triangleq x - b$. The original optimization problem becomes

$$\begin{aligned} &\min_{y} ||y|| \\ &\text{s.t. } \|\Phi y - (a - \Phi b)\| = \inf_{u} \|\Phi u - (a - \Phi b)\| \end{aligned} \tag{29}$$

This is a standard minimum norm least squares problem. The solution of y is

$$y = \Phi^{+}(a - \Phi b)$$

and thus $x = \Phi^{+}a + (I - \Phi^{+}\Phi)b$.

C. Relationship of RUV and Cartesian CS

Suppose in the 3D case, $x \triangleq [r, \dot{r}, u, \dot{u}, v, \dot{v}]'$ and $x^{(l)} \triangleq [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}, \mathsf{z}, \dot{\mathsf{z}}]$, where u and v are direction cosines (see Fig.1 of [16] for an illustration). Then x and $x^{(l)}$ have the following relationships (see also [21]):

$$x = \varphi_l(x^{(l)}) = \begin{bmatrix} \sqrt{\mathsf{x}^2 + \mathsf{y}^2 + \mathsf{z}^2} \\ (\mathsf{x}\dot{\mathsf{x}} + \mathsf{y}\dot{\mathsf{y}} + \mathsf{z}\dot{\mathsf{z}})/r \\ \mathsf{x}/r \\ (\dot{\mathsf{x}} - u\dot{r})/r \\ \mathsf{y}/r \\ (\dot{\mathsf{y}} - v\dot{r})/r \end{bmatrix} \tag{30}$$

and

$$x^{(l)} = \varphi_l^{-}(x) = \begin{bmatrix} ru \\ r\dot{u} + \dot{r}u \\ rv \\ r\dot{v} + \dot{r}v \\ rw \\ r\dot{w} + \dot{r}w \end{bmatrix} \tag{31}$$

where

$$w = \sqrt{1 - u^2 - v^2}$$

D. Relationship of Polar and Cartesian CS

Suppose in the 3D case, $x \triangleq [r, \dot{r}, b, \dot{b}, e, \dot{e}]'$ and $x^{(l)} \triangleq [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}, \mathsf{z}, \dot{\mathsf{z}}]$. Then

$$x = \varphi_l(x^{(l)}) = \begin{bmatrix} \sqrt{\mathsf{x}^2 + \mathsf{y}^2 + \mathsf{z}^2} \\ (\dot{\mathsf{x}}\cos b + \dot{\mathsf{y}}\sin b)\cos e + \dot{\mathsf{z}}\sin e \\ \arccos(\mathsf{x}/(r\cos e)) \\ (\dot{\mathsf{y}} - \dot{r}\cos e \sin b + \dot{e}\mathsf{z}\sin b)/\mathsf{x} \\ \arcsin(\mathsf{z}/r) \\ (\dot{\mathsf{z}} - \dot{r}\sin e)/(r\cos e) \end{bmatrix} \tag{32}$$

and

$$x^{(l)} = \varphi_l^{-}(x) = \begin{bmatrix} r\cos e\cos b \\ \dot{r}\cos e\cos b - \dot{e}\mathsf{z}\cos b - \dot{b}\mathsf{y} \\ r\cos e\sin b \\ \dot{r}\cos e\sin b - \dot{e}\mathsf{z}\sin b + \dot{b}\mathsf{x} \\ r\sin e \\ \dot{r}\sin e + \dot{e}r\cos e \end{bmatrix} \tag{33}$$

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