18th International Conference on Information Fusion
Washington, DC - July 6-9, 2015

Adaptive Upper-Bound Linear Mean Square Error Filter of Markovian Jump Linear Systems with Generalized Unknown Disturbances

Yuemei Qin, Yan Liang, Yanbo Yang, Quan Pan and Yanting Yang
School of Automation, Northwestern Polytechnical University
Key Laboratory of Information Fusion Technology, Ministry of Education
Xi'an 710072, P. R. China
Email: qinyuemei@mail.nwpu.edu.cn

***Abstract*—This paper presents a novel estimation problem of Markovian jump linear systems (MJLSs) with generalized unknown disturbances (GUDs) in measurements. In these systems, there exist multiple uncertainties such as Markovian switching parameters, the GUD and system noises. Here, the multi-mode complexity in original system is transformed into the randomness of parameters in new system by geometric augmentation. Then, an upper-bound linear mean square error filter (UBLF) is proposed and its existence condition is given. Meanwhile, the minimum upper-bound covariances are derived so that the minimum UBLF (MUBLF) and the corresponding optimal parameters are obtained. The numerical example shows the effectiveness of the proposed filter.**

I. Introduction

Markovian jump linear systems (MJLSs) have wide and successful applications in the real world, for example, in target tracking [1], [2], fault detection and isolation [3], [4], process control [5], and signal processing [6].

As we know, a common estimation for MJLSs is in the scope of the multiple model (MM) framework. Up to present, there are three generations of multiple model estimators as mentioned in [7] and the last two generations are for the MJLSs [8], [9], [10]. However, all three generations of MM methods have the same Gaussian assumption on the process noises and measurement noises, and such Gaussian assumption does not always hold. For example, the significant altitude changes due to the maneuver of a target bring out significant variations of radar reflections and result in the so-called 'target glint', i.e., the high-tailed non-Gaussian measurement noises [11]. The recursive calculation of the first two moments of the interested vector sometimes is sufficient in practice.

In [12], the LMMSE estimator was derived from geometric augments for the MJLS based on directly estimating $x_k 1_{\{\Theta_k=i\}}$ instead of the state x_k, where $1_{\{\cdot\}}$ is a Dirac function and Θ_k is a discrete-time Markov chain. The proposed estimator can be calculated off-line because of the gain matrices of the resultant LMMSE estimator being not data-dependent. Moreover, the error covariance matrix of the derived LMMSE estimator in [12] will converge to the unique positive-semi-definite solution of an Nn-dimensional algebraic Riccati equation under the conditions of mean square stability of the MJLS and the ergodicity of the associated Markov chain, where n is the dimension of the state vector and N is the number of states of the Markov chain [13]. Furthermore, a time-invariant (a fixed-gain matrix) LMMSE estimator was derived for MJLSs [13]. By the fact that roundoff errors in solving the above Riccati equation can cause the loss of the symmetry and positive-semi-definition of the covariance of the state prediction error, an array implementation with the better numerical robustness was developed [14]. In general, all the above estimators require that the system matrices should be deterministic. Recently, considering the stochastic coefficient matrices in the MJLS, we proposed the LMSCE estimator (i.e. LMMSE estimator for MJLS with stochastic coefficient matrices) in a recursive form according to the orthogonality principle [15]. Meanwhile, the LMMSE estimation of MJLS with randomly delayed measurements is also derived in [16].

The above methods, no matter for the Gaussian noises assumption or not, or for the deterministic coefficient matrices or stochastic coefficient matrices conditions, are only considered the situation that there is no unknown disturbance (UD) in systems. However, as much knowledge as we know, there is no research reported about the MJLSs with UDs in systems. But such case exists widely in practice as that described in our previous paper [17]. For example, in the maneuvering target tracking under electronic countermeasures (ECM), due to the sensor bias and deception jamming existing in sensor measurements [19], [20], there is additional UD to the nominal measurement model, while the target maneuver can be modeled as the first-order Markov jump among a complete model set. Therefore, it is desirable to design a filter to solve the MJLSs with UDs.

In this paper, an upper-bound LMMSE estimator (UBLF) is designed for the MJLSs with generalized UD (GUD) in the measurement equation, and the optimal parameters are derived so that the minimum UBLF (MUBLF) is obtained. The numerical example shows the effectiveness of the proposed estimator.

The rest of this paper is organized as follows. The problem formulation is presented in Section II. In Section III, the UBLF is designed and the optimal parameters are derived so that the MUBLF is obtained. A numerical example is presented in Section IV to testify the proposed method. The conclusion is supplied follows in Section V. All proofs are presented in the Appendix.

Throughout this paper, R^n is the space of n-dimensional real vector. I and O are the identity matrix and zero matrix with the proper dimension, respectively. $(\cdot)$ denotes the same content as that in the previous parenthesis. 'col' denotes the column vector. For any two square matrices A and B, $A \geq B (A > B)$ means that $A - B$ is positive semi-definite (positive definite). The symbol ':=' means definition and notation '$\otimes$' refers to the Kronecher product. An indicator function $1_{\{\Theta_k = j\}}$ will be 1 if $\Theta_k = j$ or 0 otherwise.

II. Problem Formulation

Motivated by the above consideration, we present a discrete-time MJLS with GUDs as following:

$$x_{k+1} = F_{\Theta_k} x_k + G_{\Theta_k} w_k, \tag{1}$$

$$z_k = H_{\Theta_k} x_k + A_k \delta_k + D_{\Theta_k} v_k, \tag{2}$$

where $x_k \in R^{n_x}$ and $z_k \in R^{n_z}$ represent the system state and measurement, respectively. $\{\Theta_k\}$ is a discrete-time Markov chain with finite state space $\{1, \cdots, M\}$ and transition probability matrix $P_t = [p_{ij}]$. $\pi_{j,k} := P(\Theta_k = j)$ represents the jth mode probability at instant k. $\delta_k \in R^{n_\delta}$ represents the GUD with the same statistics as that described in our previous paper [17], [18] and A_k is the known disturbance coefficient matrix. F_{Θ_k}, G_{Θ_k}, H_{Θ_k} and D_{Θ_k} are known matrices with proper dimensions, and correspond to the $F_{i,k}$, $G_{i,k}$, $H_{i,k}$ and $D_{i,k}$ in the ith mode, respectively. $w_k \in R^{n_\omega}$ and $v_k \in R^{n_v}$ are zero-mean, white noise sequences with identity covariance matrices and independent of the initial state x_0. Here, $\{w_k\}$, $\{v_k\}$ and $\{\Theta_k\}$ are independent mutually.

Remark 2.1: As shown in system (1)-(2), there coexist multiple uncertainties including Markovian parameters (Θ_k), the GUD (δ_k) and process/measurement noises (w_k and v_k). However, in many researches, there are only two uncertainties coexisting, such as the Markovian parameters and noises coexisting [12], [13], [14] or the GUD and noises coexisting [17]. Here, the Markovian parameters are multiplicative while the GUD and process/measurement noises are additional. The deeply coupling of these three types of uncertainties leads to the demand of designing an adaptive filter with the relaxed condition of existence and the easy condition of solvability by the following considerations:

- The joint state estimation and parameter identification method for dynamics, such as the famous and wide used expectation-maximization (EM) method [21], always owns an iterative process for unknown parameter optimization with a large mount of computation cost. Meanwhile, it often treats the GUD δ_k as a constant in the sliding window, otherwise, it will be hard to converge due to the high state dimension and the lack of the valid measurement dimension, which may lead to the insolubility of the EM and further makes it impossible to identify δ_k.
- The LMMSE estimator has a small amount of computation cost and it recursively compute the first and second order moments instead of the posterior density, while the first two moments would not be derived by only using the LMMSE method due to the existence of δ_k. However, the upper bound of the second-order moment of the related state can be estimated directly instead of estimating the real second-order moment, and we can still seek for the recursive calculation for the upper bound of the second-order moment, while the estimator also owns the linear structure.

Thus, according to above considerations, the main results of the proposed UBLF and MUBLF are derived as follows.

III. The Upper Bound Linear Mean Square Error Filter

Define $\xi_k := \mathrm{col}\{x_k 1_{\{\Theta_k = i\}}, i = 1, \cdots, M\}$, we have

$$x_{k+1} = \sum_{i=1}^{M} \xi_{i,k+1}, \tag{3}$$

Thus, if the optimal estimation $\hat{\xi}_{i,k+1}$ of $\xi_{i,k+1}$ is obtained, then the optimal solution $\hat{x}_{k+1}$ of x_{k+1} can be computed.

Define F_k as an $M \times M$ block matrix with its (i,j)th sub-block being $F_{j,k} 1_{\{\Theta_{k+1}=i, \Theta_k=j\}}$; G_k as an $M \times 1$ block matrix with its jth sub-block being $\left(\sum_{i=1}^{M} G_{i,k} 1_{\{\Theta_k=i\}}\right) 1_{\{\Theta_{k+1}=j\}}$; H_k as a $1 \times M$ block matrix with its jth sub-block being $H_{j,k}$; $D_k = \sum_{i=1}^{M} D_{i,k} 1_{\{\Theta_k=i\}}$. Then, the system in (1)-(2) can be rewritten as

$$\xi_{k+1} = F_k \xi_k + G_k w_k, \tag{4}$$

$$z_k = H_k \xi_k + A_k \delta_k + D_k v_k. \tag{5}$$

It is evident that the above geometry augmentation transforms the original multi-mode complexity (i.e. $\Theta_1, \cdots, \Theta_M$) in (1)-(2) into the randomness of parameters (i.e. F_k, G_k, H_k and D_k) in (4)-(5), thus the conditions of the existence and solvability of the system are easy to be satisfied.

The aim in this paper is to construct the upper bounds of the mean square error of estimation errors, which needs mild condition to guarantee the existence and only few parameters to be estimated to ensure the solvability of the filter, instead of calculating the accurate covariance, and then minimize these bounds in the pursuit of the optimal filtering parameters with the most relaxed condition.

Make following denotations: $\hat{\xi}_{k|l}$, $\hat{\xi}_{j,k|l}$ and $\hat{x}_{k|l}$ denote the LMMSE estimates of ξ_k, $\xi_{j,k}(x_k 1_{\{\Theta_k=j\}}), j = 1, \cdots, M$ and x_k given $Z_{1:l}$ (denotes the measurement sequence $\{z_1, ..., z_l\}$), respectively. $\bar{\mathcal{F}}_k$ represents an $M \times M$ block matrix with its $(i,j)^{th}$ sub-block $p_{ji} F_{j,k}$ and $\Phi_{k|l} := \left[\Phi_{ij,k|l}\right]$ with $\Phi_{ij,k|l} = E(\xi_{i,k} - \hat{\xi}_{i,k|l})(\xi_{j,k} - \hat{\xi}_{j,k|l})^T$.

Definition 3.1: *(Definition of the UBLF).* An UBLF for (4)-(5) is defined as

$$\begin{aligned}&\{\hat{\xi}_{k+1|k+1},\Phi^*_{k+1|k},S^*_{k+1},\Phi^*_{k+1|k+1}\}\\&=\mathrm{UBLF}\{\hat{\xi}_{k|k},z_{k+1},\Phi^*_{k|k},S^*_k,\Phi^*_{k|k}\},\end{aligned}\tag{6}$$

if there exists a sequence of positive-definite matrices $\Phi^*_{k+1|k}$, S^*_{k+1} and $\Phi^*_{k+1|k+1}$ that satisfy

$$\Phi^*_{k+1|k}\geq\Phi_{k+1|k}:=E(\tilde{\xi}_{k+1|k}\tilde{\xi}^T_{k+1|k}),\tag{7}$$

$$S^*_{k+1}\geq S_{k+1}:=E(\gamma_{k+1}\gamma^T_{k+1}),\tag{8}$$

$$\Phi^*_{k+1|k+1}\geq\Phi_{k+1|k+1}:=E(\tilde{\xi}_{k+1|k+1}\tilde{\xi}^T_{k+1|k+1}),\tag{9}$$

where

$$\text{state prediction error}\quad \tilde{\xi}_{k+1|k}=\xi_{k+1}-\hat{\xi}_{k+1|k},\tag{10}$$

$$\text{filter residual}\quad \gamma_{k+1}=z_{k+1}-\hat{z}_{k+1|k},\tag{11}$$

$$\text{state estimation error}\quad \tilde{\xi}_{k+1|k+1}=\xi_{k+1}-\hat{\xi}_{k+1|k+1},\tag{12}$$

and $\hat{z}_{k+1|k}$ is the prediction of measurements.

In this section, we first give the existence of positive-definite matrices (i.e. upper-bound covariances) in Definition 3.1, and then derive and provide the optimal parameters.

Theorem 3.1: *(Existence of the UBLF).* If the following two conditions are satisfied:

$$\Phi^*_{0|0}\geq\Phi_{0|0},\tag{13}$$

$$S^*_{k+1}\geq S_{k+1},\tag{14}$$

then there exist UBLFs with the following recursive upper-bound structures

$$\Phi^*_{k+1|k}=\Omega_{k+1}-\bar{\mathcal{F}}_k\Omega_k\bar{\mathcal{F}}^T_k+\bar{\mathcal{F}}_k\Phi^*_{k|k}\bar{\mathcal{F}}^T_k,\tag{15}$$

$$\begin{aligned}S^*_{k+1}&=H_{k+1}\Phi^*_{k+1|k}H^T_{k+1}+A_{k+1}\left(\varepsilon_{k+1}\Sigma\right)A^T_{k+1}\\&\quad+\sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D^T_{j,k+1}\\&=H_{k+1}\Phi^*_{k+1|k}H^T_{k+1}+\varepsilon_{k+1}A_{k+1}\Sigma A^T_{k+1}\\&\quad+\sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D^T_{j,k+1},\end{aligned}\tag{16}$$

$$\begin{aligned}\Phi^*_{k+1|k+1}&=(I-K_{k+1}H_{k+1})\Phi^*_{k+1|k}(I-K_{k+1}H_{k+1})^T\\&\quad+K_{k+1}\Big(\sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D^T_{j,k+1}\Big)K^T_{k+1},\end{aligned}\tag{17}$$

that satisfy (7)-(9), and the original state estimate and its covariance for system in (1)-(2) are

$$\hat{x}_{k+1|k+1}=\sum_{i=1}^{M}\hat{\xi}_{i,k+1|k+1},\tag{18}$$

$$P_{k+1|k+1}=\sum_{i=1}^{M}\sum_{j=1}^{M}\Phi_{ij,k+1|k+1},\tag{19}$$

where the adjust factor $\varepsilon_{k+1}\geq 0$ is a parameter to be estimated, $\Sigma>0$ is the setting matrix and requires positive-definite, $\Omega_k:=E(\xi_k\xi^T_k)=diag\{\Omega_{i,k},i=1,\cdots,M\}$ and the filter gain K_{k+1} is a function of $\Phi^*_{k+1|k}$ and S^*_{k+1}.

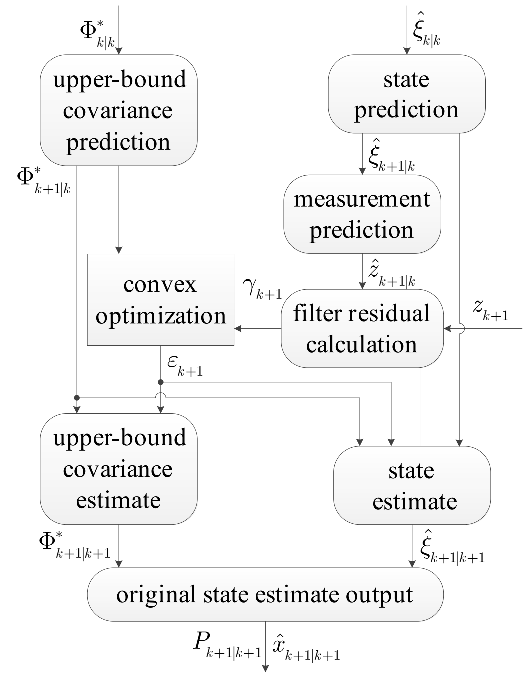

Fig. 1. The implementation of the MUBLF

Proof: See the Appendix A.

Theorem 3.2: *(Optimal Parameters).* If the initialization (13) is satisfied, and for any ε_{k+1} satisfying $S_{k+1}\leq S^*_{k+1}\big|_{\varepsilon_{k+1}}$ and any filter gain K_{k+1}, there have

$$\begin{aligned}\Phi_{k+1|k+1}\big|_{K_{k+1}}&\leq\Phi^*_{k+1|k+1}\Big|_{\varepsilon^{opt}_{k+1},K^{opt}_{k+1}}\\&\leq\Phi^*_{k+1|k+1}\Big|_{\varepsilon_{k+1},K_{k+1}},\end{aligned}\tag{20}$$

$$S_{k+1}\leq S^*_{k+1}\Big|_{\varepsilon^{opt}_{k+1}}\leq S^*_{k+1}\big|_{\varepsilon_{k+1}},\tag{21}$$

then there exist the minimum UBLF (MUBLF), i.e. the optimal solution, and the optimal filter parameters are

$$\varepsilon^{opt}_{k+1}=\min\left\{\varepsilon_{k+1}\left|\varepsilon_{k+1}\in\Lambda_{k+1}\right.\right\},\tag{22}$$

$$K^{opt}_{k+1}=\Phi^*_{k+1|k}H^T_{k+1}S^{*-1}_{k+1},\tag{23}$$

where the convex optimization (22) can be solved by the following inequalities:

$$\left\{\begin{aligned}&\varepsilon_{k+1}\geq 0\\&H_{k+1}\Phi^*_{k+1|k}H^T_{k+1}+\varepsilon_{k+1}A_{k+1}\Sigma A^T_{k+1}\\&+\sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D^T_{j,k+1}\geq S_{k+1}\end{aligned}\right.\tag{24}$$

where

$$\Lambda_{k+1}=:\left\{\varepsilon_{k+1}\left|\varepsilon_{k+1}\geq 0,S^*_{k+1}\right|_{\varepsilon_{k+1}}\geq S_{k+1}\right\},\tag{25}$$

and S_{k+1} is unknown and substituted by its unbiased estimate $\hat{S}_{k+1}=:\gamma_{k+1}\gamma^T_{k+1}$.

Proof: See the Appendix B.

Remark 3.1: After constructing the upper-bound structure of the UBLF and MUBLF in Theorems 3.1 and 3.2, it is evident that the condition of the solvability is easier to meet since they do not need to identify any parameter. What's more, minimizing the upper-bound covariances can decrease the peak error adaptively, which may be significant in target tracking systems.

According to the above derivation, the procedure of the MUBLF method is shown in Fig. 1 .

IV. Numerical Example

A numerical example for the discrete-time MJLS with GUD in measurements is presented in this section based on two Markovian states, compared with the LMMSE in [12]. The related matrices and parameters are given as follows.

Mode 1:

$$F^1 = \begin{bmatrix} 0.95 & 0.15 \\ -0.25 & 0.75 \end{bmatrix}, \quad G^1 = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix},$$
$$H^1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad D^1 = \sqrt{2} I_2.$$

Mode 2:

$$F^2 = \begin{bmatrix} 0.75 & -0.15 \\ 0.25 & 0.95 \end{bmatrix}, \quad G^2 = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix},$$
$$H^2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad D^2 = \sqrt{2} I_2$$

where, $T = 0.5s$ is the sample time and $\omega = 0.1257$.

The system evolution is simulated for 50 steps with the first 15 steps in model 1, the middle 20 steps in model 2, and the last 15 steps in model 1. The true initial state value is given to be $x_0 = [1.75 \ 2]^T$. The covariances of the process noise and measurement noise in model 1 and 2 are the same and equal to $Q = 1$ and $R = I_2$, respectively. The matrix $A = 1.5 \times [\ 1.75 \quad 1.25\ ]^T$ and the GUD is assumed to be the uniform distribution with the interval $[-3, 3]$.

In the LMMSE estimator in [12] and the proposed MUBLF, the mode switching probability matrix is $P_t = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}$, and the mode probabilities are $\pi_{1,k} = 0.9$, $\pi_{2,k} = 0.1$ for the first 15 steps, $\pi_{1,k} = 0.1$, $\pi_{2,k} = 0.9$ for the middle 20 steps, and $\pi_{1,k} = 0.9$, $\pi_{2,k} = 0.1$ for the last 15 steps. The initial state estimate is $\hat{x}_{0|0} = 0_{2\times 1}$ and the corresponding covariance is $P_{0|0} = I_2$. Meanwhile, Σ in the proposed MUBLF equals identity matrix.

The system state elements and their estimates are given in Fig. 2 by a single Monte Carlo simulation, and it shows that the estimated state by the proposed MUBLF is more accurate than that by the LMMSE in [12]. The RMSEs of both two elements in state is show in Fig. 3 via 1000 Monte Carlo simulations, which show the better estimate accuracy of the MUBLF method than that of the LMMSE. Therefore, the proposed MUBLF method is effective when dealing with the MJLSs with GUDs.

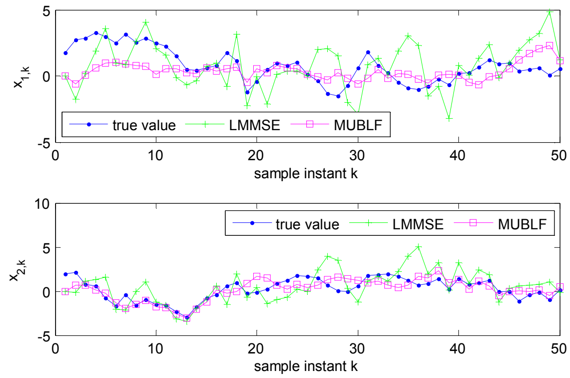

Fig. 2. True value v.s. estimated value

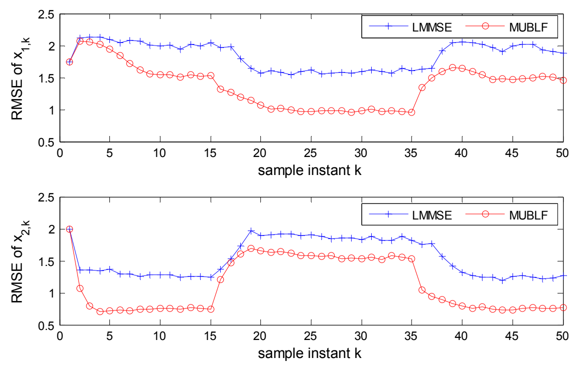

Fig. 3. RMSEs of the proposed MUBLF and LMMSE

V. Conclusion

This paper presents the state estimation problem of MJLSs with multiple uncertainties coupling of Markovian stochastic switching parameters, the GUD and system noises. Such system is transformed into a system with only randomness of parameters by geometric augmentation. In the new system, an UBLF is proposed and the existence condition is given. Then, the MUBLF and its optimal parameters are also derived. The simulation shows the effectiveness of the proposed filter.

Acknowledgment

This work is supported by the National Science Foundation Council of China under Grant 61374023, 61374159 and 61135001.

References

[1] Y. Boers, H. Driessen, "A multiple model multiple hypothesis filter for Markovian switching systems", *Automatica*, 41, pp. 709-716, 2005.

[2] W. Liu, H. Zhang, Z. Wang, "A novel truncated approximation based algorithm for state estimation of discrete-time Markov jump linear systems", *Signal Processing*, 91, pp. 702-712, 2011.

[3] L. Blackmore, S. Rajamanoharan, B. C. Williams, "Active estimation for jump Markov linear systems", *IEEE Trans. Automat. Control*, vol. 10, no. 53, pp. 2223-2236, 2008.

[4] J. Ru, X. Li, "Variable-structure multiple-model approach to fault detection, identification and estimation", *IEEE Trans. Control Syst. Technol.*, vol. 16, no. 5, pp. 1029-1038, 2008.

[5] J. Xiong, J. Lam, H. Gao, D. W. C. Ho, "On robust stabilization of Markovian jumpsystems with uncertain switching probabilities", *Automatica*, 41, pp. 897-903, 2005.

[6] L. A. Johnston, V. Krishnamurthy, "An improvement to the interacting multiple model (IMM) algorithm", *IEEE Trans. Signal Process.*, vol. 49, no. 12, pp. 2909-2923, 2001.

[7] X. Li, V. P. Jilkov, "Survey of maneuvering target tracking. Part V: multiple-model methods", *IEEE Trans. Aerosp. and Electron. Syst.*, vol. 41, no. 4, pp. 1255-1321, 2005.

[8] J. Liu, R. Li, "Hierarchical adaptive interacting multiple model algorithm", *IET Control Theory Appl.*, vol. 2, no. 6, pp. 479-487, 2008.

[9] T. Kirubarajan, Y. Bar-Shalom, K. R. Pattipati, I. Kadar, "Ground target tracking with variable structure IMM estimator", *IEEE Trans. Aerosp. and Electron. Syst.*, vol. 36, no. 1, pp. 26-46, 2000.

[10] J. Lan, X. Li, C. Mu, "Best model augmentation for variable-structure multiple-model estimation", *IEEE Trans. Aerosp. and Electron. Syst.*, vol. 47, no. 3, pp. 2008-2025, 2011.

[11] I. Bilik, J. Tabrikian, "Maneuvering target tracking in the presence of glint using the nonlinear gaussian mixture Kalman filter", *IEEE Trans. Aerosp. and Electron. Syst.*, vol. 46, no. 1, pp. 246-262, 2010.

[12] O. L. V. Costa, "Linear minimum mean square error estimation for discrete-time Markovian jump linear system", *IEEE Trans. Automat. Control*, vol. 39, no. 8, pp. 1685-1689, 1994.

[13] O. L. V. Costa, "Stationary Filter for Linear Minimum Mean Square Error Estimation for Discrete-time Markovian Jump Linear Systems ", *IEEE Trans. Automat. Control*, vol. 47, no. 8, pp. 1351-1356, 2002.

[14] M. H. Terra, J. Y. Ishihara and A. P. Junior, "Array Algorithm for Filtering of Discrete-Time Markovian Jump Linear Systems ", *IEEE Trans. Automat. Control*, vol. 52, no. 7, pp. 1293-1296, 2007.

[15] Y. Yang, Y. Liang, Q. Pan, Y. Qin and F. Yang, "Linear minimum-mean-square error estimation of Markovian jump linear systems with stochastic coefficient matrices", *IET Control Theory Appl.*, vol. 8, iss. 12, pp. 1112-1126, 2014.

[16] Y. Yang, Y. Liang, F. Yang, Y. Qin and Q. Pan, "Linear minimum-mean-square error estimation of Markovian jump linear systems with randomly delayed measurements", *IET Signal Process.*, vol. 8, iss. 6, pp. 658-667, 2014.

[17] Y. Qin, Y. Liang, Y. Yang, Z. Wang, F. Yang, "Adaptive filter of nonlinear systems with generalized unknown disturbances", *IET Radar Sonar Navig.*, vol. 4, no. 8, pp. 307-317, 2014.

[18] Y. Yang, Y. Qin, Y. Liang, Q. Pan, F. Yang, "Adaptive filter for linear systems with generalized unknown disturbance in Measurements", *16th International Conference on Information Fusion.*, Istanbul, Turkey, Jul 9-12, 2013.

[19] W. Li, H. Leung, Y. Zhou, "Space-time registration of radar and ESM using unscented Kalman filter", *IEEE Trans. Aerosp. and Electron. Syst.*, vol. 40, no. 3, pp. 824-836, 2004.

[20] B. Bao, S. Xiao, X. Wang, T. Wang, "Maximum likelihood approach to the estimation and discrimination of exoatmospheric active phantom tracks using motion features", *IEEE Trans. Aerosp. and Electron. Syst.*, vol. 48, no. 1, pp. 794-819, 2012.

[21] Z. Li, S. Chen, H. Leung, É. Bossé, "Joint data association, registration, and fusion using EM-KF", *IEEE Trans. Aerosp. and Electron. Syst.*, vol. 46, no. 2, pp. 496-507, 2010.

APPENDIX

A. The Proof of Theorem 3.1

Define the second-order moments $\Pi_{k|l} := E(\hat{\xi}_{k|l}\hat{\xi}_{k|l}^T)$ and $\Upsilon_{k|l} := E(\xi_k - \hat{\xi}_{k|l})(z_k - \hat{z}_{k|l})^T$. In Eqs. (10)-(11), we have

$$\begin{aligned}\hat{\xi}_{k+1|k} &= E(F_k\xi_k + G_k w_k|\,Z_{1:k}) \\ &= E(F_k\xi_k|\,Z_{1:k}) + E(G_k w_k|\,Z_{1:k}) \\ &= E(F_k)E(\xi_k|\,Z_{1:k}) = \bar{\mathcal{F}}_{\|}\hat{\xi}_{k|k},\end{aligned} \tag{26}$$

$$\begin{aligned}\hat{z}_{k+1|k} &= E(z_{k+1}|Z_{1:k}) \\ &= E(H_{k+1}\xi_{k+1} + A_{k+1}\delta_{k+1} + D_{k+1}v_{k+1}|\,Z_{1:k}) \\ &= H_{k+1}\hat{\xi}_{k+1|k},\end{aligned} \tag{27}$$

$$\hat{\xi}_{k+1|k+1} = \hat{\xi}_{k+1|k} + K_{k+1}\gamma_{k+1}. \tag{28}$$

In above derivation, (26) holds because $E(G_k w_k|\,Z_{1:k})$ equals zero matrix. (27) holds because $E(D_{k+1}v_{k+1}|Z_{1:k})$ equals zero matrix and the best approximation of $E(A_k\delta_{k+1}|Z_{1:k})$ is zero as δ_{k+1} is the unknown term. Then according to (11), we can obtain

$$\begin{aligned}\gamma_{k+1} &= z_{k+1} - \hat{z}_{k+1|k} \\ &= z_{k+1} - H_{k+1}\hat{\xi}_{k+1|k} \\ &= H_{k+1}\tilde{\xi}_{k+1|k} + A_{k+1}\delta_{k+1} + D_{k+1}v_{k+1}.\end{aligned} \tag{29}$$

By the fact that v_{k+1} is independent of ξ_{k+1} and $\hat{\xi}_{k+1|k}$, we obtain the independence between v_{k+1} and $\tilde{\xi}_{k+1|k}$. Further using the independence of δ_{k+1} and v_{k+1}, we have

$$\begin{aligned}S_{k+1} =& E(\gamma_{k+1}\gamma_{k+1}^T) \\ =& E(H_{k+1}\tilde{\xi}_{k+1|k} + A_{k+1}\delta_{k+1} + D_{k+1}v_{k+1})(\cdot)^T \\ =& E(H_{k+1}\tilde{\xi}_{k+1|k} + A_{k+1}\delta_{k+1})(\cdot)^T \\ &+ E(D_{k+1}v_{k+1}v_{k+1}^T D_{k+1}^T).\end{aligned} \tag{30}$$

As shown in (30), the presence of δ_{k+1} represents the uncertainty. Thus, we have

$$\begin{aligned}\mathrm{S}_{k+1} \geq& E(H_{k+1}\tilde{\xi}_{k+1|k})(\cdot)^T + E(D_{k+1}v_{k+1}v_{k+1}^T D_{k+1}^T) \\ =& E(H_{k+1}\tilde{\xi}_{k+1|k}\tilde{\xi}_{k+1|k}^T H_{k+1}^T) \\ &+ E(D_{k+1}v_{k+1}v_{k+1}^T D_{k+1}^T) \\ =& H_{k+1}\Phi_{k+1|k}H_{k+1}^T + \sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D_{j,k+1}^T,\end{aligned} \tag{31}$$

where

$$\begin{aligned}\Phi_{k+1|k} &= E(\xi_{k+1} - \hat{\xi}_{k+1|k})(\xi_{k+1} - \hat{\xi}_{k+1|k})^T \\ &= E(\xi_{k+1}\xi_{k+1}^T) - E(\hat{\xi}_{k+1|k}\hat{\xi}_{k+1|k}^T) \\ &= diag\{\Omega_{i,k+1}, i = 1, \cdots, M\} - \Pi_{k+1|k},\end{aligned} \tag{32}$$

with

$$\begin{aligned}\Omega_{i,k+1} &= E(\xi_{i,k+1}\xi_{i,k+1}^T) \\ &= E(F_k^i\xi_k + G_k^i w_k)(F_k^i\xi_k + G_k^i w_k)^T \\ &= E(F_k^i\xi_k\xi_k^T(F_k^i)^T) + E(G_k^i w_k w_k^T(G_k^i)^T) \\ &= \sum_{j=1}^{M}p_{ji}F_{j,k}\Omega_{j,k}F_{j,k}^T + \sum_{j=1}^{M}p_{ji}\pi_{j,k}G_{j,k}G_{j,k}^T,\end{aligned} \tag{33}$$

$$\begin{aligned}\Pi_{k+1|k} &= E(\hat{\xi}_{k+1|k}\hat{\xi}_{k+1|k}^T) = E(\bar{\mathcal{F}}_k\hat{\xi}_{k|k}\hat{\xi}_{k|k}^T\bar{\mathcal{F}}_k^T) \\ &= \bar{\mathcal{F}}_k\Pi_{k|k}\bar{\mathcal{F}}_k^T,\end{aligned} \tag{34}$$

$$\begin{aligned}\Pi_{k+1|k+1} &= E(\hat{\xi}_{k+1|k+1}\hat{\xi}_{k+1|k+1}^T) \\ &= E(\hat{\xi}_{k+1|k} + \Upsilon_{k+1|k}S_{k+1}^{-1}\gamma_{k+1|k})(\cdot)^T \\ &= E(\hat{\xi}_{k+1|k}\xi_{k+1|k}^T) + (\Upsilon_{k+1|k}S_{k+1}^{-1})S_{k+1}(\cdot)^T \\ &= \Pi_{k+1|k} + \Upsilon_{k+1|k}S_{k+1}^{-1}\Upsilon_{k+1|k}^T,\end{aligned} \tag{35}$$

where F_k^i and G_k^i are the ith row entry of F_k, G_k, respectively. Thus, (8) and (16) hold.

In (35), $\Upsilon_{k+1|k}$ is represented as follows.

$$\begin{aligned}\Upsilon_{k+1|k} &= E(\tilde{\xi}_{k+1|k}\gamma_{k+1|k})\\ &=E(\tilde{\xi}_{k+1|k}(H_{k+1}\tilde{\xi}_{k+1|k} + A_{k+1}\delta_{k+1} + D_{k+1}v_{k+1})^T)\\ &=\Phi_{k+1|k}H_{k+1}^T.\end{aligned} \tag{36}$$

According to above derivation, we have

$$\Phi_{k+1|k} = \Omega_{k+1} - \bar{\mathcal{F}}_k\Omega_k\bar{\mathcal{F}}_k^T + \bar{\mathcal{F}}_k\Phi_{k|k}\bar{\mathcal{F}}_k^T, \tag{37}$$

and

$$\begin{aligned}\Phi_{k+1|k+1} &= \Omega_{k+1} - \Pi_{k+1|k+1}\\ &=\Omega_{k+1} - (\Pi_{k+1|k} + \Upsilon_{k+1|k}S_{k+1}^{-1}\Upsilon_{k+1|k}^T)\\ &=\Phi_{k+1|k} - \Upsilon_{k+1|k}S_{k+1}^{-1}\Upsilon_{k+1|k}^T\\ &=(I - K_{k+1}H_{k+1})\Phi_{k+1|k}(I - K_{k+1}H_{k+1})^T\\ &\quad + K_{k+1}(\sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D_{j,k+1}^T)K_{k+1}^T.\end{aligned} \tag{38}$$

As the first condition (13) of Theorem 3.2, and we assume $\Phi_{k|k}^* \geq \Phi_{k|k}$. Using mathematical induction and (37), we can have

$$\Phi_{k+1|k}^* - \Phi_{k+1|k} = \bar{\mathcal{F}}_k\left(\Phi_{k|k}^* - \Phi_{k|k}\right)\bar{\mathcal{F}}_k^T. \tag{39}$$

As $\bar{\mathcal{F}}_k$ is a full rank square matrix, we have

$$\Phi_{k+1|k}^* - \Phi_{k+1|k} \geq 0. \tag{40}$$

So, from (37) and (40), we can obtain (7) and (15).

Putting (7) into (17) yields

$$\begin{aligned}\Phi_{k+1|k+1}^* &= (I - K_{k+1}H_{k+1})\Phi_{k+1|k}^*(I - K_{k+1}H_{k+1})^T\\ &\quad + K_{k+1}(\sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D_{j,k+1}^T)K_{k+1}^T\\ &\geq (I - K_{k+1}H_{k+1})\Phi_{k+1|k}(I - K_{k+1}H_{k+1})^T\\ &\quad + K_{k+1}(\sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D_{j,k+1}^T)K_{k+1}^T\\ &=\Phi_{k+1|k+1}.\end{aligned} \tag{41}$$

Thus, from (40) and (41), (9) and (17) can be obtained.

B. The Proof of Theorem 3.2

According to Theorem 3.1, the set $\left\{\varepsilon_{k+1}\left|S_{k+1}^* \geq S_{k+1}\right.\right\}$ will not be empty if an UBLF exists. Because $S_{k+1}^*\left|_{\varepsilon_{k+1,1}}\right. \leq S_{k+1}^*\left|_{\varepsilon_{k+1,2}}\right.$ when $\varepsilon_{k+1,1} \leq \varepsilon_{k+1,2}$, so if $\varepsilon_{k+1,1} \in \Lambda_{k+1}$, then must have $\varepsilon_{k+1,2} \in \Lambda_{k+1}$. Therefore,

$$\begin{aligned}\Lambda_{k+1} &= \left\{\varepsilon_{k+1}\left|\varepsilon_{k+1} \geq 0, S_{k+1}^*\left|_{\varepsilon_{k+1}}\right. \geq S_{k+1}\right.\right\}\\ &= \left\{\varepsilon_{k+1}\left|\varepsilon_{k+1} \geq 0\right.\right\} \cap \left\{\varepsilon_{k+1}\left|S_{k+1}^*\left|_{\varepsilon_{k+1}}\right. \geq S_{k+1}\right.\right\}\end{aligned} \tag{42}$$

is not null. Hence there exists $\varepsilon_{k+1}^{opt} = \min\left\{\varepsilon_{k+1}\left|\varepsilon_{k+1} \in \Lambda_{k+1}\right.\right\}$.

Then, it is only necessary to testify that ε_{k+1}^{opt} and K_{k+1}^{opt} guarantee (20)-(21). It is easy to know $\varepsilon_{k+1} \geq \varepsilon_{k+1}^{opt}$ to any $\varepsilon_{k+1} \in \Lambda_{k+1}$ for the definition of ε_{k+1}^{opt}. Thus

$$\begin{aligned}\Delta S_{k+1} &= S_{k+1}^*\left|_{\varepsilon_{k+1}}\right. - S_{k+1}^*\Big|_{\varepsilon_{k+1}^{opt}}\\ &=(H_{k+1}\Phi_{k+1|k}^*H_{k+1}^T + \varepsilon_{k+1}A_{k+1}\Sigma A_{k+1}^T\\ &\quad + \sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D_{j,k+1}^T)\\ &\quad - (H_{k+1}\Phi_{k+1|k}^*H_{k+1}^T + \varepsilon_{k+1}^{opt}A_{k+1}\Sigma A_{k+1}^T\\ &\quad + \sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D_{j,k+1}^T)\\ &=(\varepsilon_{k+1} - \varepsilon_{k+1}^{opt})A_{k+1}\Sigma A_{k+1}^T \geq 0\end{aligned} \tag{43}$$

$$\begin{aligned}\Delta\Phi_{k+1|k+1}^{-1} &= \Phi_{k+1|k+1}^{*-1}\Big|_{\varepsilon_{k+1},K_{k+1}} - \Phi_{k+1|k+1}^{*-1}\Big|_{\varepsilon_{k+1}^{opt},K_{k+1}}\\ &=\Phi_{k+1|k}^{*-1} + H_{k+1}^T(\varepsilon_{k+1}A_{k+1}\Sigma A_{k+1}^T + R_{k+1})H_{k+1}\\ &\quad - \Phi_{k+1|k}^{*-1} + H_{k+1}^T(\varepsilon_{k+1}^{opt}A_{k+1}\Sigma A_{k+1}^T + R_{k+1})H_{k+1}\\ &=(\varepsilon_{k+1} - \varepsilon_{k+1}^{opt})H_{k+1}^T A_{k+1}\Sigma A_{k+1}^T H_{k+1} \geq 0.\end{aligned} \tag{44}$$

When deriving (43), both the expression of S_{k+1}^* in (16) and the equation (45) are used. The equation (45) holds due to the factor ε_{k+1} not included in (15). To obtain (44), the other form of the covariance expression in (46) is used.

$$\begin{aligned}&\Phi_{k+1|k}^*\Big|_{\varepsilon_{k+1}} - \Phi_{k+1|k}^*\Big|_{\varepsilon_{k+1}^{opt}}\\ &=(\Omega_{k+1} - F_k\Omega_k\bar{F}_k^T + F_k\Phi_{k|k}^*\bar{F}_k^T)\Big|_{\varepsilon_{k+1}}\\ &\quad - (\Omega_{k+1} - F_k\Omega_k\bar{F}_k^T + F_k\Phi_{k|k}^*\bar{F}_k^T)\Big|_{\varepsilon_{k+1}^{opt}}\\ &=O,\end{aligned} \tag{45}$$

$$\begin{aligned}\Phi_{k+1|k+1}^{*-1} &= \Phi_{k+1|k}^{-1} + H_{k+1}^T(\varepsilon_{k+1}A_{k+1}\Sigma A_{k+1}^T\\ &\quad + \sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D_{j,k+1}^T)H_{k+1}.\end{aligned} \tag{46}$$

From (44), we can easy to know

$$\begin{aligned}&\Delta\Phi_{k+1|k+1}\\ &=\Phi_{k+1|k+1}^*\left|_{\varepsilon_{k+1},K_{k+1}}\right. - \Phi_{k+1|k+1}^*\Big|_{\varepsilon_{k+1}^{opt},K_{k+1}} \geq 0.\end{aligned} \tag{47}$$

Through (16) and $\sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D_{j,k+1}^T > 0$, $S_{k+1}^* \geq \sum_{j=1}^{M}\pi_{j,k+1}D_{j,k+1}D_{j,k+1}^T > 0$ exists, so the symmetric and positive definite matrix $S_{k+1}^*\Big|_{\varepsilon_{k+1}^{opt}}$ can be represented by $G_{k+1}G_{k+1}^T$, where G_{k+1} is full rank. Suppose the optimal gain K_{k+1}^{opt} exists, and then its existence needs to testify. After transformation, there is

$$\begin{aligned}&\Phi_{k+1|k+1}^{*}\Big|_{\varepsilon_{k+1}^{opt},K_{k+1}}\\&=\Phi_{k+1|k}^{*}\Big|_{\varepsilon_{k+1}^{opt}}-B_{k+1}B_{k+1}^{T}\\&\quad+(K_{k+1}G_{k+1}-B_{k+1})(K_{k+1}G_{k+1}-B_{k+1})^{T},\end{aligned}\tag{48}$$

where $B_{k+1}=\Phi_{k+1|k}^{*}\Big|_{\varepsilon_{k+1}^{opt}}H_{k+1}^{T}G_{k+1}^{-T}$.

By the fact $(K_{k+1}G_{k+1}-B_{k+1})(K_{k+1}G_{k+1}-B_{k+1})^{T}\geq O$, we have $\Phi_{k+1|k+1}^{*}\geq\Phi_{k+1|k}^{*}\Big|_{\varepsilon_{k+1}^{opt}}-B_{k+1}B_{k+1}^{T}$ if and only if $K_{k+1}^{opt}=B_{k+1}G_{k+1}^{-1}=\Phi_{k+1|k}^{*}\Big|_{\varepsilon_{k+1}^{opt}}H_{k+1}^{T}S_{k+1}^{*-1}$. So, there is

$$\begin{aligned}\Phi_{k+1|k+1}^{*}\Big|_{\varepsilon_{k+1}^{opt},K_{k+1}}\geq&\Phi_{k+1|k}^{*}\Big|_{\varepsilon_{k+1}^{opt}}-B_{k+1}B_{k+1}^{T}\\=&\Phi_{k+1|k+1}^{*}\Big|_{\varepsilon_{k+1}^{opt},K_{k+1}^{opt}}.\end{aligned}\tag{49}$$

Thus, an MUBLF exists and $\varepsilon_{k+1}^{opt}\in\Lambda_{k+1}$. Thus, (7)-(9) are obtained and (20)-(21) are further obtained based on (7)-(9) and (43), (47), (49).