18th International Conference on Information Fusion
Washington, DC - July 6-9, 2015

Track Association Using Augmented State Estimates

Chee-Yee Chong
Independent Researcher
Los Altos, CA U.S.A.
cychong@ieee.org

Shozo Mori
Systems & Technology Research
Sunnyvale, CA U.S.A.
shozo.mori@stresearch.com

Abstract* - *Track association has not received as much attention as track fusion in distributed multi-sensor multi-target tracking, especially for targets whose motion models involve process noise. One exception is an association metric that uses the cross-covariance of the track state estimates at a single time. For track fusion, it has been shown that the centralized state estimate can be obtained by fusion of augmented state estimates consisting of state estimates at multiple times. Association using augmented state estimates is even more natural because the association likelihood should consider the entire state trajectory of a track, and not just the estimates at the last time. Starting with a general association likelihood function, we show that augmented states allow exact evaluation of the track association likelihood. For problems involving Gaussian densities, the association metric is the standard Mahalanobis or chi-square metric with the single time state estimate replaced by the augmented state estimate. Simulations compare the performance of association using augmented state estimates of different lengths and the method using cross-covariances. Results demonstrate excellent performance for augmented state association even when the full augmented state is not used and filtered estimates instead of smoothed estimates are used.

Keywords: association likelihood, augmented state estimates, cross-covariance association, non-zero process noise, track association, track fusion

1 Introduction

In multi-sensor multi-target tracking, a centralized processing architecture can produce the best performance because the processing site can utilize all the sensor measurements without loss of information due to intermediate processing. However, centralized processing is not always practical because communicating measurements require high communication bandwidth. Thus, most real world tracking systems utilize a hierarchical or distributed architecture with local tracking at the sensors and processing of tracks from multiple sensors at a fusion site [1], [2].

In a distributed tracking architecture, the fusion site has to perform two main functions. The first function is associating local tracks from the sensors to form global tracks corresponding to the same targets. The second function is fusing the state estimates for the associated local tracks to form estimates of the global tracks.

The first function is frequently called track fusion even though strictly speaking track fusion should include both track association and track state estimate fusion. Fusing track state estimates is more challenging than fusing sensor measurements because the state estimation errors may be dependent due to past communication or common process noise. An overview of available track fusion techniques can be found in [3], [4]. The most common approach [5]–[7], frequently called tracklet fusion or equivalent measurement, uses the current and predicted local estimates to find the new information received since the last fusion time. However, it does not generate the optimal global estimate when process noise is present and the fusion rate is lower than the sensor observation rate. Optimality can be regained if the state is augmented to be the entire state trajectory for all observation times since the most recent fusion [8]–[10]. Reference [11] contains the detailed equations for tracklet fusion with augmented state and shows that an augmented state with very short length, e.g., at the last two or three times, produces almost optimal results.

Other track fusion algorithms address dependence that can be characterized by cross-sensor covariances of the local estimation errors. Examples include maximum likelihood estimate [12], [13], best linear unbiased estimate (BLUE) [14], [15], and minimum variance (MV) estimate [16], [17]. These approaches require computation and communication of cross covariances and the fused estimate may not be globally optimal because it is only the best estimate given the local estimates characterized by local and cross-sensor estimation error covariances.

Track association has to address the same dependence issues in the local tracks. In particular, computation of the association likelihood or metric between two tracks has to account for the dependence in the track state estimation errors. As compared with track fusion, little has been published on association for tracks with dependent state estimation errors. Reference [18] uses the cross-covariance of [12] to compute the association metric between two tracks.

Association using augmented state estimates is intuitive because association should consider the track state estimates at multiple times and not just the most recent state estimates. It was suggested in [19] and further developed in [20]. Performance evaluation in [16], [3], [4] shows much better performance than association with state estimates at a single time, even when cross-covariance is used. Despite good performance of association with augmented state estimates, it has not received much attention because communicating the full augment state estimates may not be practical.

There is resurgent interest in augmented state estimation because efficient algorithms have been developed to compute the augmented state estimate and its covariance. Furthermore, accumulated state density (ASD), which is another name for augmented state, can be used in estimation problems such as out of sequence measurement processing [21]. In [11], [22], it is shown that augmented state fusion can produce an estimate that is almost optimal for very short state lengths.

This paper provides a review of track association with augmented state estimates and evaluates its performance for different track lengths and sampling times. We will develop the augmented state association metric for a general problem and the specific equations for the Gaussian case. The numerical examples will show that good association performance can be obtained without using the full augmented state. Furthermore, performance degradation is minimal when filtered estimates are used instead of smoothed estimates.

The rest of this paper is structured as follows. Section 2 presents the track association problem and a general form for the track association likelihood. Section 3 reviews augmented state estimation for Gaussian problems. Section 4 derives the association metric using augmented states and presents the cross-covariance association metric that will be used for comparison. Section 5 presents numerical results that demonstrate the performance of association with augmented state estimates. Section 6 contains conclusions.

2 Association problem and solution

Track association is the first step in distributed multi-sensor multi-target tracking. The local tracks from the individual sensors have to be associated to form global tracks before their state estimates can be fused to compute the state estimates of the global tracks.

2.1 Track association problem

For simplicity, we consider a multi-target tracking problem with two sensors. Each sensor observes the target states and processes the measurements to generate tracks with state estimates. Each track consists of a sequence of measurements hypothesized to be from the same target, so that τ_i^1, track i from sensor 1, has measurements $Z^{1i} = (z_k^{1i})_{k=1}^K$, and τ_j^2, track j from sensor 2, has measurements $Z^{2j} = (z_k^{2j})_{k=1}^K$. The measurements z_k^{1i} and z_k^{2j} are collected at synchronous times $t_k, k = 1,...,K$, but this assumption can be removed without much difficulty. Given a track, the probability densities of the state at any time or the state estimates and error covariances can be computed from the associated measurements.

We assume that the measurements in each track are correctly associated with a target. The track association problem or track-to-track association problem for two sensors is to find the pairs of tracks (τ_i^1, τ_j^2) that originate from the same targets.

2.2 Maximum a posteriori probability association

The best association hypothesis can be found as the hypothesis that maximizes the posterior probability of the association hypothesis. The general form for the probability is first introduced in [19] and generalized in [23]. In this paper, we assume perfect detection with no false alarms. Thus, for n targets, the two sensors will produce two sets of tracks $(\tau_i^1)_{i=1}^n$ and $(\tau_j^2)_{j=1}^n$. We further assume that n, the number of targets, is a Poisson random variable, and that the target states are modeled as independent identically distributed random processes. The measurements z_k^{si} are conditionally independent given the target state x_k^i.

The association hypothesis is a permutation a on the set $\{1,...,n\}$ that maps a track of sensor 1 to a track of sensor 2. It can be shown that the posterior probability of the association hypothesis a is given by

$$p(a) = C^{-1} \prod_{i=1}^{n} l(i, a(i)) \tag{1}$$

where C is a normalizing constant and $l(i, j)$ is the likelihood of associating τ_i^1 and τ_j^2 with the same target. Furthermore, the track association likelihood can be expressed as

$$l(i, j) = p(Z^{1i}, Z^{2j}) \tag{2}$$

The best association hypothesis can be found as the permutation $\hat{a}$ that maximizes the probability $p(a)$ in (1). If we define the track association metric $L(i, j)$ as

$$L(i, j) = \ln l(i, j) \tag{3}$$

then the best association hypothesis is found by minimizing the association cost

$$J(a) = \sum_{i=1}^{n} L(i, a(i)) \tag{4}$$

This is a bipartite assignment problem that can be solved by many efficient algorithms [24], [25].

2.3 Track association likelihood

The key to track association is computing the track association likelihood $l(i, j)$ given two tracks τ_i^1 and τ_j^2. Let x be a random "state" of the target that causes Z^{1i} and Z^{2j} to be conditionally independent given x, i.e.,

$$p(Z^{1i}, Z^{2j} \mid x) = p(Z^{1i} \mid x) p(Z^{2j} \mid x) \quad (5)$$

Then,

$$\begin{aligned} p(Z^{1i}, Z^{2j}) &= \int p(Z^{1i}, Z^{2j} \mid x) p(x) dx \\ &= \int p(Z^{1i} \mid x) p(Z^{2j} \mid x) p(x) dx \\ &= p(Z^{1i}) p(Z^{2j}) \int \frac{p(x \mid Z^{1i}) p(x \mid Z^{2j})}{p(x)} dx \end{aligned} \quad (6)$$

According to (6), the likelihood of associating two tracks depends on the similarity between the conditional probability densities of the target "state" for the two tracks. In fact the integral in (6) is the normalizing constant for computing the conditional probability of the "state" x given the combined measurements (Z^{1i}, Z^{2j}), i.e.,

$$p(x \mid Z^{1i}, Z^{2j}) = \frac{p(Z^{1i}) p(Z^{2j})}{p(Z^{1i}, Z^{2j})} \frac{p(x \mid Z^{1i}) p(x \mid Z^{2j})}{p(x)} \quad (7)$$

Thus track association likelihood calculation and distributed estimation [26] are closely related. Since the terms $p(Z^{1i}) p(Z^{2j})$ in (6) do not affect the overall optimization problem, they can be left out in the track association likelihood computation.

For deterministic target dynamics, when there is no process noise, the target state x_K at the last measurement time t_K satisfies the conditional independence assumption (5). Thus, the track association likelihood can be computed from the conditional probability densities $p(x_K \mid Z^{si})$, $s = 1, 2$, $i = 1, .., n$, i.e.,

$$l(i, j) = p(Z^{1i}) p(Z^{2j}) \int \frac{p(x_K \mid Z^{1i}) p(x_K \mid Z^{2j})}{p(x_K)} dx_K \quad (8)$$

For non-deterministic dynamics, when the process noise is non-zero, the cumulative measurements Z^{si}, $s = 1, 2$, $i = 1, .., n$, are no longer conditionally independent given x_K. However, they are conditionally independent given the augmented state $X_K = (x_k)_{k=1}^{K}$, consisting of the target states at all measurement times. Thus, the track association likelihood is computed as

$$l(i, j) = p(Z^{1i}) p(Z^{2j}) \int \frac{p(X_K \mid Z^{1i}) p(X_K \mid Z^{2j})}{p(X_K)} dX_K \quad (9)$$

where $p(X_K \mid Z^{si})$ is the posterior density of the augmented state X_K.

3 Association for Gaussian problems

Since the track association likelihoods in Section II are developed for general target and sensor models, their computation requires integrals on probability densities. The computation can be simplified significantly if the probabilities are Gaussian. Let $p(x)$, $p(x \mid Z^{1i})$, and $p(x \mid Z^{2j})$ be Gaussians with means $\bar{x}$, $\hat{x}_{1i}$, and $\hat{x}_{2j}$, and covariances $\bar{P}$, P_{1i}, and P_{2j}.

3.1 Metric of similarity with fused estimate

This section sketches the derivation of the metric that is introduced in [3], [4], [16], [19] without derivation. Let $p(x \mid Z^{1i}, Z^{2j})$ be a Gaussian density with mean $\hat{x}_F$ and covariance P_F. From (6) and (7),

$$\begin{aligned} &\int \frac{p(x \mid Z^{1i}) p(x \mid Z^{2j})}{p(x)} dx = \frac{p(Z^{1i}, Z^{2j})}{p(Z^{1i}) p(Z^{2j})} \\ &= \frac{p(x \mid Z^{1i}) p(x \mid Z^{2j})}{p(x) p(x \mid Z^{1i}, Z^{2j})} \\ &= \left(\frac{|P_F| |\bar{P}|}{|P_{1i}| |P_{2j}|} \right)^{1/2} \\ &\exp(-1/2) \begin{pmatrix} \| x - \hat{x}_{1i} \|^2_{P_{1i}^{-1}} + \| x - \hat{x}_{2j} \|^2_{P_{2j}^{-1}} - \\ \| x - \hat{x}_F \|^2_{P_F^{-1}} - \| x - \bar{x} \|^2_{\bar{P}^{-1}} \end{pmatrix} \end{aligned} \quad (10)$$

If we set

$$x - \hat{x}_{1i} = (x - \hat{x}_F) + (\hat{x}_F - \hat{x}_{1i}) \quad (11)$$

$$x - \hat{x}_{2j} = (x - \hat{x}_F) + (\hat{x}_F - \hat{x}_{2j}) \quad (12)$$

$$x - \bar{x} = (x - \hat{x}_F) + (\hat{x}_F - \bar{x}) \quad (13)$$

and use the fusion equations [26]

$$P_F^{-1} = P_{1i}^{-1} + P_{2j}^{-1} - \bar{P}^{-1} \quad (14)$$

$$P_F^{-1} \hat{x}_F = P_{1i}^{-1} \hat{x}_{1i} + P_{2j}^{-1} \hat{x}_{2j} - \bar{P}^{-1} \bar{x} \quad (15)$$

then (10) becomes

$$\begin{aligned} &\int \frac{p(x \mid Z^{1i}) p(x \mid Z^{2j})}{p(x)} dx = \left(\frac{|P_F| |\bar{P}|}{|P_{1i}| |P_{2j}|} \right)^{1/2} \\ &\exp(-1/2) \begin{pmatrix} \| \hat{x}_F - \hat{x}_{1i} \|^2_{P_{1i}^{-1}} + \| \hat{x}_F - \hat{x}_{2j} \|^2_{P_{2j}^{-1}} - \\ \| \hat{x}_F - \bar{x} \|^2_{\bar{P}^{-1}} \end{pmatrix} \end{aligned} \quad (16)$$

Since the determinant terms cancel out in the optimization, the track association metric, which is the logarithm of (16), becomes

$$L(i,j) = \| \hat{x}_F - \hat{x}_{1i} \|^2_{P_{1i}^{-1}} + \| \hat{x}_F - \hat{x}_{2j} \|^2_{P_{2j}^{-1}} - \| \hat{x}_F - \bar{x} \|^2_{\bar{P}^{-1}} \tag{17}$$

This metric evaluates the association likelihood by comparing the fused track state estimate with the individual estimates of the tracks associated to form the fused track.

3.2 Metric of similarity between local estimates

This metric, first introduced in [27], is the Mahanolobis or chi-squared distance commonly used for track association. It can be derived by assuming no prior $p(x)$ for the target state. Then the track association likelihood becomes

$$l(i,j) = p(Z^{1i})p(Z^{2j}) \int p(x \mid Z^{1i}) p(x \mid Z^{2j}) dx \tag{18}$$

From the Appendix,

$$\int p(x \mid Z^{1i}) p(x \mid Z^{2j}) dx = g(0; \hat{x}_{1i} - \hat{x}_{2j}, P_{1i} + P_{2j}) \tag{19}$$

where $g(x;m,P) = |(2\pi)^n P|^{-1/2} \exp(-1/2)(x-m)^T P^{-1}(x-m)$ is the Gaussian density function with mean m and covariance P. Thus the track association metric is

$$L(i,j) = (\hat{x}_{1i} - \hat{x}_{2j})^T (P_{1i} + P_{2j})^{-1} (\hat{x}_{1i} - \hat{x}_{2j}) \tag{20}$$

The state estimates and error covariances in (20) may be for the state at a given time or the augmented state consisting of states at multiple times. The two metrics (17) and (20) are similar but (20) is more convenient because it depends only on the local track estimates and does not require track state fusion to generate $\hat{x}_F$.

3.3 Cross-covariance track association metric

This association metric addresses the dependence in target tracks due to process noise by using the cross-covariance between the track state estimates of the two sensors at a single time [18]. It is included in this paper because its performance will be compared with the augmented state association metric.

Let P_{12ij} be the cross-covariance between the state estimates $\hat{x}_{1i}$ and $\hat{x}_{2j}$. Then the cross-covariance association metric is

$$L(i,j) = (\hat{x}_{1i} - \hat{x}_{2j})^T (P_{1i} + P_{2j} - P_{12ij} + P_{12ij}^T)^{-1} (\hat{x}_{1i} - \hat{x}_{2j}) \tag{21}$$

Since the cross-covariance is only evaluated at the end time of the track, this metric does not utilize all the information in the tracks. Thus, its performance is expected to be better than (20) evaluated at a single time but worse than the augmented state association metric that exploits information at multiple times. Section 5 will demonstrate that this is true.

4 Augmented state estimation

This section presents the equations for calculating the augmented state estimate and error covariances.

4.1 State and measurement models

The state to be estimated is modeled by the linear system

$$x_{k+1} = F_k x_k + G_k w_k \tag{22}$$

where $x_k \in R^n$ is the state at time t_k with $k = 0,1,2,\ldots$, F_k and G_k are matrices representing the system dynamics, and w_k is a zero-mean white random process with covariance Q_k.

For each sensor s, the measurement z_k^s at time t_k is

$$z_k^s = H_k^s x_k + v_k^s \tag{23}$$

where H_k^s is the measurement matrix, and v_k^s is a zero-mean white noise process with covariance R_k^s. The measurement and process noises are assumed to be independent white noise processes. The initial state x_0 is independent of the noises with mean $\bar{x}_0$ and covariance $\bar{P}_0$.

4.2 Local Kalman filter

Let $Z_l^s = (z_j^s)_{j=1}^l$ be the cumulative measurements of sensor s at time t_k, $x_{k|l}^s$ and $P_{k|l}^s$ be the optimal estimate of x_k and its error covariance given Z_l^s. The local estimate $x_{k|k}^s$ and error covariance $P_{k|k}^s$ are computed by the usual equations.

Prediction

$$x_{k|k-1}^s = F_{k-1} x_{k-1|k-1}^s \tag{24}$$

$$P_{k|k-1}^s = F_{k-1} P_{k|k-1}^s F_{k-1}{}^T + G_{k-1} Q_{k-1} G_{k-1}{}^T \tag{25}$$

Update

$$(P_{k|k}^s)^{-1} x_{k|k}^s = (P_{k|k-1}^s)^{-1} x_{k|k-1}^s + i_k^s \tag{26}$$

$$(P^s_{k|k})^{-1} = (P^s_{k|k-1})^{-1} + I^s_k \tag{27}$$

with initial conditions $x^s_{0|0}$ and $P^s_{0|0}$, $i^s_k \triangleq (H^s_k)^T (R^s_k)^{-1} z^s_k$ and $I^s_k \triangleq (H^s_k)^T (R^s_k)^{-1} H^s_k$.

4.3 Augmented state filter

Let $X_k = (x_j)^k_{j=1}$ be the augmented state at time t_k. The estimate $X^s_{k|k} = [(x^s_{k|k})^T, .., (x^s_{1|k})^T]^T$ of the augmented state X_k given the measurements Z^s_k, and its error covariance $\mathbf{P}^s_{k|k}$ are computed by the following prediction and update equations [11].

Prediction

$$X^s_{k|k-1} = \begin{bmatrix} F_{k-1} x^s_{k-1|k-1} \\ X^s_{k-1|k-1} \end{bmatrix} \tag{28}$$

$$\mathbf{P}^s_{k|k-1} = \begin{bmatrix} P^s_{k|k-1} & \mathbf{F}_{k-1} \mathbf{P}^s_{k-1|k-1} \\ \mathbf{P}^s_{k-1|k-1} \mathbf{F}^T_{k-1} & \mathbf{P}^s_{k-1|k-1} \end{bmatrix} \tag{29}$$

where $\mathbf{F}_{k-1} = \left[F_{k-1}, 0_{n \times n(k-1)} \right]$ and the initial conditions are $X^s_{1|0} = x^s_{1|0}$ and $\mathbf{P}^s_{0|0} = P^s_{0|0}$.

Update

$$(\mathbf{P}^s_{k|k})^{-1} X^s_{k|k} = (\mathbf{P}^s_{k|k-1})^{-1} X^s_{k|k-1} + J_k i^s_k \tag{30}$$

$$(\mathbf{P}^s_{k|k})^{-1} = (\mathbf{P}^s_{k|k-1})^{-1} + J_k I^s_k J^T_k \tag{31}$$

where $J_k = [I_n, 0_{n \times n(k-1)}]^T$ is a $nk \times n$ matrix that selects the x_k in X_k to generate the measurement z^s_k.

These equations are similar in form to the local accumulated state densities (ASD) in [22]. However, the augmented state estimates are locally optimal whereas ASD are not locally optimal estimates because they use the relaxed evolution model.

Equations (28) – (31) will be used in computing the augmented state track association metric of (20).

5 Numerical example

This section presents a numerical example to show the benefit of track association using augmented states.

5.1 Simulation scenario

The target moves according to the 2 dimensional Ornstein-Uhlenbeck model used in [4], [11], with noise parameter $q = 2\beta\sigma^2_{VEL}$ and

$$F_k \triangleq \exp\left(A\Delta t\right);\ A \triangleq \begin{bmatrix} 0 & I_2 \\ 0 & -\beta I_2 \end{bmatrix} \tag{32}$$

$$G_k Q_k G^T_k \triangleq \int_0^{\Delta t} e^{A\tau} \begin{bmatrix} 0 & 0 \\ 0 & qI_2 \end{bmatrix} e^{A^T \tau} d\tau \tag{33}$$

The initial condition at t_0 is $P_0 = \mathrm{diag}[\sigma^2_{POS} I_2, \sigma^2_{VEL} I_2]$.

We assume two sensors that observe the position of a target, i.e., $H^s_k = [I_2, 0_{2\times 2}]$ for $s = 1, 2$ at time t_k, with $t_{k+1} - t_k = \Delta t.$ The sensors have error covariances $R^1_k = \mathrm{diag}[4, 1]$ and $R^2_k = \mathrm{diag}[1, 4]$. The complementary nature of the sensors implies that each sensor can associate its measurements into high quality local tracks but there is association uncertainty between the tracks from the two sensors.

Other simulation parameters are $\Delta t = 1$, $K = 10$, $\sigma_{POS} = 10$, $\sigma_{VEL} = 3$, and $\beta = 1$. The white noise intensity q is varied to simulate different levels of non-deterministic dynamics.

Monte Carlo analysis is conducted to generate the results. For each run, the association metrics are computed and an assignment algorithm is used to associate the local tracks. The probability of correct association, defined as the probability of each track from sensor 1 being assigned to the "correct" track (as defined by the ground truth) from sensor 2, is computed as the number of correctly associated targets over the total number of targets. This probability is then averaged over all simulation runs for a particular noise intensity.

5.2 Numerical Results

The numerical results compare the association performance using augmented state estimates and the state estimate at the last time with cross-covariance metric.

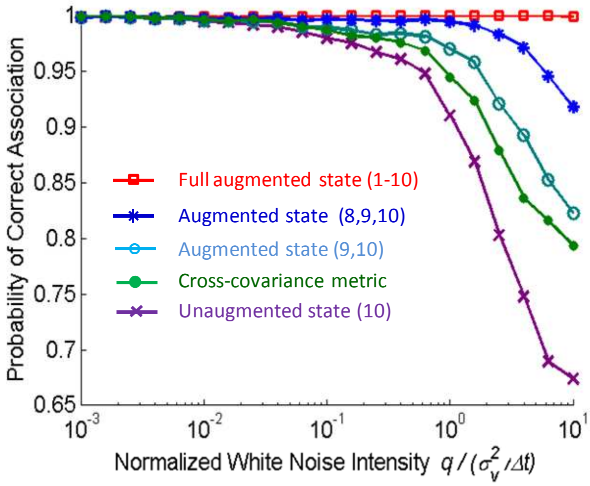

Figure. 1. Augmented state association that uses estimates of states at multiple times performs better than association with cross-covariance metric.

Figure 1 shows that association with full augmented state has excellent performance even for high process noise. When only the state estimates at the most recent two times (10,9) or three times (10,9,8) are used, augmented state association still performs better than association with the cross-covariance metric. As expected, association with unaugmented state has the worst performance.

When communication constraints do not allow the use of full augmented states, association performance can be optimized by selecting the appropriate augmented states. In particular, sampling the states at regular intervals produces better performance than using the states at the most recent times. Fig. 2 shows that sampling the states at the first and last times (1,10), or the first, middle, and last times (1,6,10), produces association performance similar to that of full augmented state, and much better than using the last two or three states. Future research will investigate the optimal number of samples and the sampling times.

In both Figures 1 and 2, the association metric (20) is computed using covariance matrices that involve cross-covariances between the estimation errors at different times, i.e., with terms such as $\mathrm{cov}(x_k - x^1_{k|K}, x_{k'} - x^2_{k'|K})$ for $k \neq k'$. Fig. 3 shows the association performance when the cross-covariances for different times are set to zero, i.e., the metric is computed using only the error covariances at the sampling times, i.e., only $\mathrm{cov}(x_k - x^1_{k|K}, x_k - x^2_{k|K})$. Since there is hardly any difference between Fig. 2 and Fig. 3, the cross-covariances do not have to be communicated, thus reducing the bandwidth requirement.

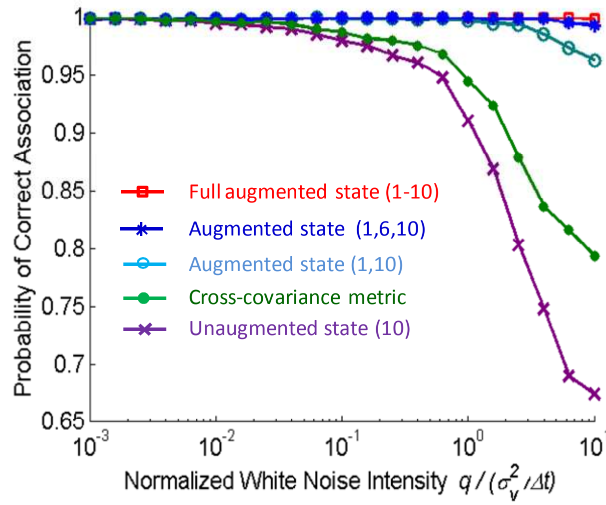

Figure 2. Augmented state association performs well by sampling at critical times

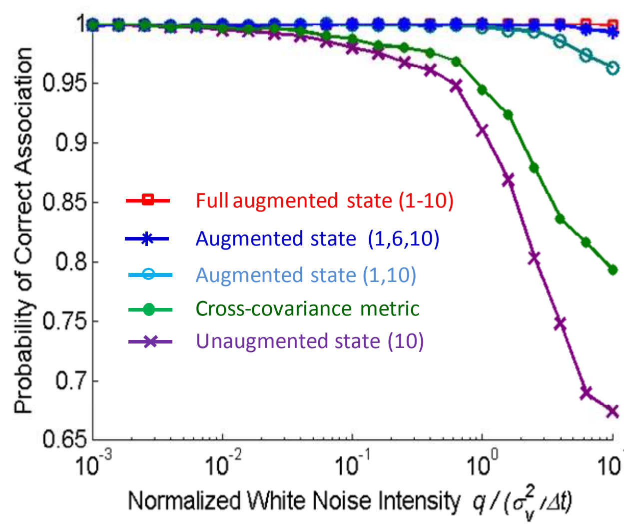

Figure 3. Good association performance with sampled augmented states can be obtained without using cross-covariances

Note that the sampled augmented state estimate is not the same as the filtered state estimate at the sampled times. Figure 4 shows that using filtered state estimates instead of smoothed estimates results in very little performance degradation. Since filtered estimates for augmented states are computed easily by local processors, track association using augmented states is practical for real world systems.

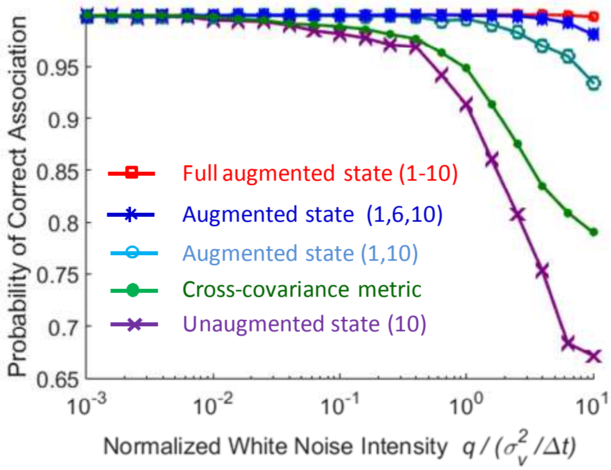

Figure 4. Filtered estimates can be used in place of smoothed estimates with minimal performance degradation

6 Conclusions

Track association for targets with non-deterministic dynamics is an important problem that has not received as much attention as track fusion. Association with augmented state estimates can be justified theoretically and performs better than other methods. However, implementation requires more communication bandwidth that association with the state estimates at a single time. We show that communication may not be an issue because good association performance does not need communication of full augmented state estimates. In fact, numerical results show that only the state estimates at two or three critical times are needed. Furthermore, excellent association performance can be obtained by using filtered estimates for the augmented states and ignoring the cross-covariances for estimates at different times.

Appendix: Integral of product of Gaussians

Let $p_1(x) = g(x; m_1, P_1)$ and $p_2(x) = g(x; m_2, P_2)$ be the densities of independent Gaussian random vectors x_1 and x_2. Consider the random vector $z = x_1 - x_2$. The probability density of z is given by

$$p_z(z) = \int_x p_1(z+x) p_2(x) dx \quad \text{(A1)}$$

Since z is a Gaussian random vector with mean $m_1 - m_2$ and covariance $P_1 + P_2$,

$$p_z(x) = g(x; m_1 - m_2, P_1 + P_2) \quad \text{(A2)}$$

Thus

$$\begin{aligned} &\int_x g(x; m_1, P_1) g(x; m_2, P_2) dx \\ &= g(0; m_1 - m_2, P_1 + P_2) \\ &= (2\pi)^{-n/2} | P_1 + P_2 |^{-1/2} \exp[-\tfrac{1}{2}(m_1 - m_2)^T (P_1 + P_2)^{-1} (m_1 - m_2)] \end{aligned} \quad \text{(A3)}$$

References

[1] M. E. Liggins II, C. Y. Chong, I. Kadar, M. G. Alford, V. Vannicola, S. Thomopoulos, “Distributed fusion architectures and algorithms for target tracking,” *Proc. IEEE,* vol. 15, no. 1, pp. 95–107, Jan. 1997.

[2] C. Y. Chong, S. Mori, W. H. Barker, and K. C. Chang, “Architectures and algorithms for track association and fusion,” *IEEE AES Magazine*, vol. 15, no. 1, pp. 5–13, Jan. 2000.

[3] S. Mori, K. C. Chang, and C. Y. Chong, “Comparison of track fusion rules and track association metrics,” *Proc. 15th Int. Conf. on Information Fusion*, Singapore, 2012.

[4] S. Mori, K. C. Chang, and C. Y. Chong, “Essence of distributed target tracking – track fusion and track association,” D. Hall, C. Y. Chong, J. Llinas, and M. Liggins II, editors, *Distributed Data Fusion for Network*-Centric Operations, CRC Press, 2012.

[5] C. Y. Chong, “Hierarchical estimation,” *Proc. MIT/ONR Workshop on C3*, Monterey, CA, 1979.

[6] O. Drummond, “Tracklets and a hybrid fusion with process noise,” *Proc. SPIE,* vol. 3163, 1997.

[7] X. Tian and Y. Bar-Shalom, “Exact algorithms for four track-to-track fusion configurations: all you wanted to know but were afraid to ask,” *Proc. 12th Int. Conf. on Information Fusion,* Seattle, USA, 2009.

[8] C. Y. Chong and S. Mori, “Graphical models for nonlinear distributed estimation,” *Proc. 7th Int. Conf. on Information Fusion*, Stockholm, Sweden, 2004.

[9] C. Y. Chong, K. C. Chang, and S. Mori, “Fundamentals of distributed estimation and tracking,” *Proc. SPIE*, vol. 8392, 2012.

[10] C. Y. Chong, K. C. Chang, and S. Mori, “Fundamentals of distributed estimation,” in D. Hall, C. Y. Chong, J. Llinas, and M. Liggins II, editors, *Distributed Data Fusion for Network-Centric Operations,* CRC Press, 2012.

[11] C. Y. Chong, S. Mori, F. Govaers, and W. Koch, “Comparison of tracklet fusion and distributed Kalman filter for track fusion,” *Proc. 17th Int. Conf. on Information Fusion*, Salamanca, Spain, 2014.

[12] Y. Bar-Shalom, and L. Campo, “The effects of the common process noise on the two-sensor fused-track covariance,” *IEEE Trans. Aerospace and Electronic Syst.,* vol. 22, no. 6, pp. 803–805, Nov. 1986.

[13] K. C. Chang, R. K. Saha, and Y. Bar-Shalom, “On optimal track-to-track fusion,” *IEEE Trans. on Aerospace and Electronic Syst.,* vol. 33, no. 4, pp. 1271–1276, Oct. 1997.

[14] Y. Zhu, and X. R. Li, "Best linear unbiased estimation fusion," *Proc. 2nd Int. Conf. on Information Fusion*. Sunnyvale, CA, 1999.
[15] X. R. Li, Y. Zhu, J. Wang, and C. Han, "Optimal linear estimation fusion – part I: unified fusion rules," *IEEE Trans. on Information Theory*, vol. 49, no. 9, pp. 2192–2208, Sep. 2003.
[16] S. Mori, W. H. Barker, C. Y. Chong, and K. C. Chang, "Track association and track fusion with non-deterministic target dynamics," *IEEE Trans. on Aerospace and Electronic Syst.* vol. 38, no. 2, pp 659–668, Apr. 2002.
[17] K. C. Chang, T. Zhi, S. Mori, and C. Y. Chong, "Performance evaluation for MAP state estimate fusion," *IEEE Trans. on Aerospace and Electronic Syst.,* vol. 40, no. 2, pp. 706–714, Apr. 2004.
[18] Y. Bar-Shalom, "On the track-to-track correlation problem," *IEEE Trans. on Automatic Control*, vol. 26, no. 2, pp. 571–572, April 1981.
[19] C. Y. Chong, S. Mori, and K. C. Chang, "Distributed multitarget multisensor tracking," in Y. Bar-Shalom, editor, *Multitarget Multisensor Tracking: Advanced Applications*, Norwood MA: Artech House, 1990, Chapter 8, pp. 247 – 295.
[20] S. Mori, K. A. Dimitri, W. H. Barker, and R. N. Lineback, "A theoretical foundation of data fusion - generic track association metric," *Proc. 7th Joint Service Data Fusion Symposium,* Laurel MD, Oct. 1994, pp. 585 – 594.
[21] W. Koch and F. Govaers, "On accumulated state densities with applications to out-of-sequence meaurement processing," *IEEE Trans. Aerospace and Electronic Syst.*, vol. 47, no. 4, pp. 2766–2778, Oct. 2011.
[22] W. Koch and F. Govaers, "On decorrelated track-to-track fusion based on accumulated state densities," *Proc. 17th Int. Conf. on Information Fusion,* Salamanca, Spain, 2014.
[23] S. Mori and C. Y. Chong, "Track-to-track association metric - I.I.D.-non-Poisson cases." *Proc. 6th Int. Conf. on Information Fusion*, Cairns, Australia, 2003.
[24] H. W. Kuhn, "The Hungarian method for the assignment problem," *Naval Research Logistics Quarterly*, pp. 83–97, 1955.
[25] J. Munkres, "Algorithms for the assignment and transportation problems", *Journal of the Society for Industrial and Applied Mathematics*, vol. 5, no. 1, pp:32–38, March 1957.
[26] C. Y. Chong, K. C. Chang, and S. Mori, "Fundamentals of distributed estimation," in D. Hall, C. Y. Chong, J. Llinas, and M. Liggins II, editors, *Distributed Data Fusion for Network-Centric Operations*, CRC Press, 2012.
[27] R. A. Singer and A. J. Kanyuck, "Computer control of multiple site track correlation," *Automatica*, vol. 7, no. 4, pp. 455–463, July 1971.