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Cyclic Bayesian Cramér-Rao Bound for Filtering in Circular State Space

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***Abstract*—Mean-squared-error (MSE) lower bounds are widely used for performance analysis in stochastic filtering problems. In many problems of this type, the nature of part of the unknown state parameters is circular or periodic. In this case, we are interested in the modulo-T estimation errors and not in the plain error values. Thus, the MSE risk and conventional MSE bounds are inappropriate for periodic stochastic filtering problems. A commonly used risk for periodic parameter estimation is the mean-cyclic-error (MCE). In this paper, we derive a cyclic version of the Bayesian Cramér-Rao bound (BCRB) on the MCE of any recursive filter. The performance of the cyclic BCRB is evaluated for phase tracking and compared to the MCEs of existing filters.**

***Index Terms*—Mean-squared-error (MSE) lower bounds, periodic stochastic filtering, mean-cyclic-error (MCE), Bayesian Cramér-Rao bound (BCRB), phase tracking**

I. Introduction

Stochastic filtering problems are frequently encountered in many fields such as signal processing, communications, and control. The mean-squared-error (MSE) risk is widely used for performance analysis of stochastic filtering algorithms and various algorithms attempt to achieve minimum MSE (MMSE) performance. The well known Kalman filter [1] is commonly-used for stochastic filtering in linear dynamic systems due to its tractability and the fact that for linear models it results in the linear MMSE estimator, that coincides with the MMSE estimator for Gaussian noise. For the general case of nonlinear/non-Gaussian estimation problem, suboptimal stochastic filtering methods, such as the extended Kalman filter (e.g. [2]–[5]), the unscented Kalman filter (e.g. [6], [7]), and the particle filter (e.g. [8]–[10]), are adopted.

The MMSE estimator may be intractable in the general nonlinear/non-Gaussian case and assessing the optimal performance may be difficult. In this case, computationally manageable MSE lower bounds that can be calculated recursively are very useful for stochastic filtering performance analysis and feasibility study. In [11], a recursive computation of the Bayesian Cramér-Rao bound (BCRB) for stochastic filtering problems is derived. Recursive computations of tighter MSE bounds, including the Bayesian Bhattacharyya, Bobrovsky-Zakai, and Weiss-Weinstein bounds, are derived in [12]. In [13], a combined Bayesian Cramér-Rao/Weiss-Weinstein bound is derived for tracking target bearing. Recursive computations of the Weiss-Weinstein lower bound for a class of Markovian dynamic systems and for hybrid continuous and discrete random states are derived in [14] and [15], respectively.

In many stochastic filtering problems, the unknown parameters have a circular or periodic nature, for example, phase and frequency [11], [16], [17] as well as direction-of-arrival (DOA) [18]–[20]. In a T-periodic stochastic filtering problem, the modulo-T estimation error is of interest. Thus, in such problems the appropriate risk is based on a T-periodic cost function [21]–[28] and traditional Bayes risks, such as the MSE, are inappropriate.

The mean-cyclic-error (MCE) [23]–[26], [29], [30] and the mean-squared-periodic-error (MSPE) [26]–[28] are the most commonly used risks for periodic stochastic filtering problems. The squared-periodic-error (SPE) cost function is the square of the modulo-T estimation error, which is not a differentiable function. Thus, minimization of the MSPE risk and implementation of corresponding performance bounds, as developed in [28], may become intractable. In contrast to the SPE, the cyclic-error (CE) cost function is a smooth periodic function of the estimation error and is suitable for the derivation of tractable optimal estimators and performance bounds. Thus, in this paper we adopt the MCE risk.

Improved recursive algorithms for periodic stochastic filtering that utilize the MCE and MSPE risks for performance analysis, are suggested in [16], [19], [25], [31]–[37] based on angular distributions and circular statistics [29], [38]. Lower bounds can be very useful for performance evaluation of these periodic stochastic filters. However, as mentioned above, MSE lower bounds are inappropriate for periodic stochastic filtering problems. Therefore, in such problems, lower bounds on periodic risks can be useful. For off-line periodic estimation problems, few Bayesian lower bounds have been suggested. Scalar Bayesian lower bounds on general periodic risks are derived in [23] via a modification of the Ziv-Zakai lower bound. A new class of Bayesian lower bounds on the MCE is derived in [24].

In this paper, we extend the cyclic BCRB from [24], derived for off-line parameter estimation, to periodic stochastic filtering. In particular, we derive a recursive implementation of the cyclic BCRB, which is the cyclic version of the recursive implementation of the BCRB from [11]. It is shown that the cyclic BCRB for periodic stochastic filtering has less restrictive assumptions than the conventional BCRB. For example, for uniform distribution of the initial state parameter, the cyclic BCRB may exist, while the BCRB does not (e.g. [39]). Finally, the performance of the cyclic BCRB is evaluated for phase tracking problem and compared to the MCE performance of the Kurz-Gilitschenski-Hanebeck (KGH) filter from [37], the Willsky-Lo (WL) filter from [25], [31], and the particle filter from [8].

The remainder of the paper is organized as follows. In Section II, we formulate the periodic stochastic filtering setup. The properties of the MCE and recursive estimation methods that attempt to approximate the MMCE estimator, are reviewed in Section III. The cyclic BCRB for periodic stochastic filtering is derived in Section IV, based on extension of the class of MCE bounds from [24] for stochastic filtering problems. Evaluation of the proposed cyclic BCRB is performed for phase tracking problem in Section V. Our conclusions appear in Section VI.

II. Periodic stochastic filtering setup

A. Notations

In the sequel, we denote vectors by boldface lowercase letters and matrices by boldface uppercase letters. The mth element of the vector $\mathbf{b}$ and the (m,q)th element of the matrix $\mathbf{B}$ are denoted by b_m and $[\mathbf{B}]_{m,q}$, respectively. Given a scalar function g dependent on a vector $\boldsymbol{\theta}$, its gradient w.r.t. $\boldsymbol{\theta}$, denoted as $\frac{\mathrm{d}g(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}}$, is a row vector, in which the qth element equals to $\frac{\partial g(\boldsymbol{\theta})}{\partial \theta_q}$. The (m,q)th element of the Hessian matrix of g w.r.t. $\boldsymbol{\theta}$ is its second-order derivative w.r.t. θ_m and θ_q denoted as $\frac{\partial^2 g(\boldsymbol{\theta})}{\partial \theta_m \partial \theta_q}$. An $M \times 1$ vector and an $M \times Q$ matrix with zero entries are denoted by $\mathbf{0}_M$ and $\mathbf{0}_{M\times Q}$, respectively. The notations $\mathbf{A} \succeq \mathbf{B}$ and $\mathbf{A} \succ \mathbf{B}$ imply that $\mathbf{A} - \mathbf{B}$ is a positive-semidefinite matrix and a positive-definite matrix, respectively, where $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices of the same size. The notations $\log(\cdot)$, $|\cdot|$, and $\angle\cdot$ stand for the natural logarithm, the absolute value, and the phase of a complex scalar, respectively. We assume that the phase of a complex scalar is restricted to the interval $[-\pi,\pi)$. The operators $(\cdot)^T$ and $(\cdot)^H$ denote transpose and conjugate transpose, respectively and $j \triangleq \sqrt{-1}$. The modulo-2π operator, which maps $\rho \in \mathbb{R}$ to $[-\pi,\pi)$, is denoted by $[\rho]_{2\pi} = \rho - 2\pi \left\lfloor \frac{1}{2} + \frac{\rho}{2\pi} \right\rfloor$, where $\lfloor\cdot\rfloor$ is the floor operator. The operators of expectation and conditional expectation given an event Z, are denoted as $\mathrm{E}[\cdot]$ and $\mathrm{E}[\cdot|Z]$, respectively.

B. State and observation models

Consider the nonlinear discrete-time stochastic filtering problem in which, at each time step $n = 1, 2, \ldots$, we are interested in estimating the periodic state parameter θ_n, supported on $\Omega_\theta = [-\frac{T}{2}, \frac{T}{2})$. For simplicity, it is assumed that $T = 2\pi$. Extension of the results obtained in this paper for an arbitrary period is straightforward. The periodic state parameter θ_n is evolving according to the state model

$$\theta_{i+1} = a(\theta_i, \mathbf{w}_i), \; i = 0, 1, 2, \ldots, \tag{1}$$

where $a : \Omega_\theta \times \mathbb{R}^P \to \Omega_\theta$ is the state transition function and $\{\mathbf{w}_i\}$ is a sequence of mutually independent noise vectors that are independent of past and present states. It is assumed that $\mathbf{w}_i$ has a known probability density function (pdf), $\forall i \geq 0$. The measurement vector $\mathbf{x}_i \in \mathbb{C}^N$ is obtained at each time step based on the current state, according to the observation equation

$$\mathbf{x}_i = \mathbf{h}(\theta_i, \boldsymbol{\nu}_i), \; i = 1, 2, \ldots, \tag{2}$$

where $\mathbf{h} : \Omega_\theta \times \mathbb{C}^Q \to \mathbb{C}^N$ is the measurement function and $\{\boldsymbol{\nu}_i\}$ is a sequence of mutually independent noise vectors with known pdfs that are independent of past and present states and the state noise. The initial state parameter θ_0 is assumed to have a known *a-priori* pdf f_{θ_0}. We denote by $\Omega_{\mathbf{x}^{(n)}}$ and $\Omega_{\boldsymbol{\theta}^{(n)}}$, the nth step measurement and state spaces, where $\mathbf{x}^{(n)} \triangleq \left[\mathbf{x}_1^T, \ldots, \mathbf{x}_n^T\right]^T$ and the unknown state parameter vector $\boldsymbol{\theta}^{(n)} \triangleq [\theta_0, \ldots, \theta_n]^T$. The Hilbert space of absolutely square integrable scalar functions w.r.t. the joint distribution of $\mathbf{x}^{(n)}$ and $\boldsymbol{\theta}^{(n)}$ is denoted by $\mathcal{L}_2(\Omega_{\mathbf{x}^{(n)}} \times \Omega_{\boldsymbol{\theta}^{(n)}})$. By using Bayes rule, it can be verified (see e.g. [11]) that up to time step n, the joint pdf of $\mathbf{x}^{(n)}$ and $\boldsymbol{\theta}^{(n)}$, is given by

$$\begin{aligned} &f_{\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}}\left(\boldsymbol{\beta}^{(n)}, \boldsymbol{\alpha}^{(n)}\right) = \\ &f_{\theta_0}(\alpha_0) \prod_{i=1}^{n} f_{\theta_i|\theta_{i-1}}(\alpha_i|\alpha_{i-1}) \prod_{i=1}^{n} f_{\mathbf{x}_i|\theta_i}(\boldsymbol{\beta}_i|\alpha_i), \end{aligned} \tag{3}$$

where $f_{\theta_i|\theta_{i-1}}$ and $f_{\mathbf{x}_i|\theta_i}$ are the conditional pdfs obtained from (1) and (2), respectively. In addition, we define

$$f_{\theta_0}^{(\mathrm{p})}(\alpha_0) \triangleq \sum_{l=-\infty}^{\infty} f_{\theta_0}(\alpha_0 + 2\pi l), \tag{4}$$

$$\begin{aligned} &f_{\theta_i|\theta_{i-1}}^{(\mathrm{p})}(\alpha_i|\alpha_{i-1}) \triangleq \\ &\sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_{\theta_i|\theta_{i-1}}(\alpha_i + 2\pi l|\alpha_{i-1} + 2\pi m), \end{aligned} \tag{5}$$

$\forall i = 1, \ldots, n$, and

$$f_{\mathbf{x}_i|\theta_i}^{(\mathrm{p})}(\boldsymbol{\beta}_i|\alpha_i) \triangleq \sum_{m=-\infty}^{\infty} f_{\mathbf{x}_i|\theta_i}(\boldsymbol{\beta}_i|\alpha_i + 2\pi m), \tag{6}$$

$\forall i = 1, \ldots, n$, which are 2π-periodic extensions, w.r.t. $\{\theta_i\}_{i=0}^n$, of f_{θ_0}, $f_{\theta_i|\theta_{i-1}}$, and $f_{\mathbf{x}_i|\theta_i}$, respectively. By using (4)-(6), we obtain the 2π-periodic extension of $f_{\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}}$ w.r.t. $\boldsymbol{\theta}^{(n)}$:

$$\begin{aligned} &f_{\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}}^{(\mathrm{p})}\left(\boldsymbol{\beta}^{(n)}, \boldsymbol{\alpha}^{(n)}\right) \triangleq \\ &f_{\theta_0}^{(\mathrm{p})}(\alpha_0) \prod_{i=1}^{n} f_{\theta_i|\theta_{i-1}}^{(\mathrm{p})}(\alpha_i|\alpha_{i-1}) \prod_{i=1}^{n} f_{\mathbf{x}_i|\theta_i}^{(\mathrm{p})}(\boldsymbol{\beta}_i|\alpha_i). \end{aligned} \tag{7}$$

The estimation error of the nth state parameter, θ_n, is denoted as

$$\varepsilon = \hat{\theta}_n\left(\mathbf{x}^{(n)}\right) - \theta_n, \tag{8}$$

where $\hat{\theta}_n : \Omega_{\mathbf{x}^{(n)}} \to \Omega_\theta$ is an estimator of θ_n based on $\mathbf{x}^{(n)}$. For simplicity of notations, in the following $f_{\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}}$ and $f^{(\mathrm{p})}_{\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}}$ will be replaced by f_n and $f_n^{(\mathrm{p})}$, respectively.

III. MCE RISK

A. Background

The MCE risk, which is commonly used in circular statistics, is given by (see e.g. [25], [29]):

$$\mathrm{MCE}\left(\hat{\theta}\right) = 2 - 2\mathrm{E}\left[\cos\left(\hat{\theta} - \theta\right)\right], \tag{9}$$

for an estimator $\hat{\theta}$ of an unknown random parameter θ. It can be verified that the MCE of $\hat{\theta}$ is equal to the MSE of the estimator $e^{j\hat{\theta}}$ of the complex parameter $e^{j\theta}$, i.e.

$$\mathrm{MCE}\left(\hat{\theta}\right) = \mathrm{E}\left[\left|e^{j\hat{\theta}} - e^{j\theta}\right|^2\right]. \tag{10}$$

In addition, by substituting $\delta = \hat{\theta} - \theta$ in the following inequality

$$2 - 2\cos(\delta) \le \delta^2, \ \forall \delta \in \mathbb{R}, \tag{11}$$

it can be verified that the MCE of an estimator of θ is always smaller than or equal to its MSE. By substituting $\delta = \hat{\theta} - \theta$ in the equality

$$\lim_{\delta \to 0}\left\{\frac{2 - 2\cos(\delta)}{\delta^2}\right\} = 1, \tag{12}$$

it can be verified that for $\hat{\theta} - \theta \to 0$ the MCE risk coincides with the MSE risk.

In the Bayesian framework, optimal estimation is based on minimization of a given Bayes risk. In this case, it can be verified that the MMCE estimator, given a measurement vector $\mathbf{x}$, is (see e.g. [25], [29, pp. 21-22], [31]):

$$\hat{\theta}_{\mathrm{MMCE}}(\mathbf{x}) \triangleq \begin{cases} \angle \mathrm{E}\left[e^{j\theta}|\mathbf{x}\right], & \mathrm{E}\left[e^{j\theta}|\mathbf{x}\right] \neq 0 \\ 0, & \text{otherwise} \end{cases}. \tag{13}$$

It can be shown that the minimum MCE, which is obtained by the MMCE estimator, is equal to [24], [25]

$$\mathrm{MCE}\left(\hat{\theta}_{\mathrm{MMCE}}\right) = 2 - 2\mathrm{E}\left[\left|\mathrm{E}\left[e^{j\theta}\middle|\mathbf{x}\right]\right|\right]. \tag{14}$$

B. Recursive estimation methods

In this subsection, we review the recursive filters that will be assessed in this paper: the KGH filter, the WL filter, and the particle filter.

1) KGH filter: The KGH filter, derived in [37], provides a general framework for the estimation of a circular state based on the wrapped normal and the von Mises distributions, which are approximated with wrapped Dirac mixture distributions. It can be used for state estimation of circular systems with nonlinear system and measurement functions. This filter relies on deterministic sampling techniques as described in [35], [40]–[42]. For this filter, the nth step estimator with D deterministic samples, denoted as $\hat{\theta}^{(D)}_{\mathrm{KGH},n}$, is calculated based on a wrapped Dirac mixture distribution that approximates the nth step *a-posteriori* pdf $f_{\theta_n|\mathbf{x}^{(n)}}$, from which the MMCE term in (13) can be calculated.

2) WL filter: The recursive WL filter [25], [31] attempts to compute the MMCE estimator from (13) at each time step. It is restricted to the following state model:

$$\theta_{i+1} = [\theta_i + w_i]_{2\pi}, \ i = 0, 1, 2, \ldots, \tag{15}$$

where the state noise $\{w_i\}$ is supported on Ω_θ with pdf f_{w_i} and its periodic extension $f^{(\mathrm{p})}_{w_i}$. In this method, it is also assumed that the pdf periodic extensions $f^{(\mathrm{p})}_{\theta_0}$, $f^{(\mathrm{p})}_{w_i}$, and $f^{(\mathrm{p})}_{\mathbf{x}_i|\theta_i}(\mathbf{x}_i|\cdot)$, can be represented via Fourier series with Fourier coefficients $\left\{c_l^{(0)}\right\}$, $\left\{r_l^{(i)}\right\}$, and $\left\{d_l^{(i)}(\mathbf{x}_i)\right\}$, respectively.

Under this model, it can be shown that the MMCE estimator of the state at the nth step, denoted as $\hat{\theta}_{\mathrm{MMCE},n}$, is given by

$$\hat{\theta}_{\mathrm{MMCE},n}\left(\mathbf{x}^{(n)}\right) \triangleq \begin{cases} \angle c_{-1}^{(n|n)}\left(\mathbf{x}^{(n)}\right), & c_{-1}^{(n|n)}\left(\mathbf{x}^{(n)}\right) \neq 0 \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, 2, \ldots, \tag{16}$$

where $c_l^{(i|k)}\left(\mathbf{x}^{(k)}\right)$ is the lth Fourier coefficient of the periodic extension of the *a-posteriori* pdf, $f^{(\mathrm{p})}_{\theta_i|\mathbf{x}^{(k)}}\left(\cdot|\mathbf{x}^{(k)}\right)$. In fact, at each time step n the recursive WL filter computes the Fourier coefficients of the *a-posteriori* pdf $f^{(\mathrm{p})}_{\theta_n|\mathbf{x}^{(n)}}\left(\cdot|\mathbf{x}^{(n)}\right)$, as presented in the following.

Initial prediction:

$$c_l^{(1|0)} = 2\pi c_l^{(0)} r_l^{(0)}, \ \forall l \in \mathbb{Z}. \tag{17}$$

Initial estimation:

$$c_l^{(1|1)}\left(\mathbf{x}^{(1)}\right) = \frac{\gamma_l^{(1)}\left(\mathbf{x}^{(1)}\right)}{2\pi\gamma_0^{(1)}\left(\mathbf{x}^{(1)}\right)}, \ \forall l \in \mathbb{Z}, \tag{18}$$

where

$$\gamma_l^{(i)}\left(\mathbf{x}^{(i)}\right) \triangleq \sum_{m=-\infty}^{\infty} c_m^{(i|i-1)}\left(\mathbf{x}^{(i-1)}\right) d_{l-m}^{(i)}(\mathbf{x}_i).$$

The ith step prediction:

$$c_l^{(i|i-1)}\left(\mathbf{x}^{(i-1)}\right) = 2\pi c_l^{(i-1|i-1)}\left(\mathbf{x}^{(i-1)}\right) r_l^{(i-1)}, \quad \forall l \in \mathbb{Z}, \ i = 2, 3, \ldots. \tag{19}$$

The ith step estimation:

$$c_l^{(i|i)}\left(\mathbf{x}^{(i)}\right)=\frac{\gamma_l^{(i)}\left(\mathbf{x}^{(i)}\right)}{2\pi\gamma_0^{(i)}\left(\mathbf{x}^{(i)}\right)},\ \forall l\in\mathbb{Z},\ i=2,3,\ldots. \quad (20)$$

In practice, the MMCE estimator is approximated by a finite number of Fourier coefficients that are usually computed numerically. We denote the approximated MMCE estimator with L Fourier coefficients as $\hat{\theta}^{(L)}_{\mathrm{WL},n}$.

3) Particle filter: The sequential importance resampling particle filter is derived in [8]. For this filter, The nth step estimator with S particles, denoted as $\hat{\theta}^{(S)}_{\mathrm{PAR},n}$, is calculated based on the particle filter probability mass function, which approximates the nth step *a-posteriori* pdf $f_{\theta_n|\mathbf{x}^{(n)}}$, to calculate the MMCE estimate in (13).

IV. Cramér-Rao-type MCE lower bound for periodic stochastic filtering

A. General class of MCE lower bounds for periodic stochastic filtering

In this subsection, the class of MCE bounds from [24] is extended for periodic stochastic filtering problems. The following theorem presents a general class of lower bounds on the MCE of any estimator of the state parameter θ_n.

Theorem 1: Let $\mathbf{v}_n : \Omega_{\mathbf{x}^{(n)}}\times\Omega_{\boldsymbol{\theta}^{(n)}}\to\mathbb{C}^M$ be an arbitrary auxiliary vector function whose elements belong to $\mathcal{L}_2(\Omega_{\mathbf{x}^{(n)}}\times\Omega_{\boldsymbol{\theta}^{(n)}})$ and its second-order moment matrix is defined as

$$\mathbf{B}_n\triangleq\mathrm{E}\left[\mathbf{v}_n\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)\mathbf{v}_n^H\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)\right]. \quad (21)$$

It is assumed that $\mathbf{v}_n$ satisfies

C.1) $\mathbf{B}_n\succ\mathbf{0}_{M\times M}$.

C.2) There exists a constant vector (independent of $\mathbf{x}^{(n)}$ and $\boldsymbol{\theta}^{(n)}$), $\mathbf{k}_n\in\mathbb{C}^M$, $0<\left|\left|\mathbf{k}_n\right|\right|<\infty$, s.t.

$$\mathrm{E}\left[e^{-j\theta_n}\mathbf{v}_n\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)\middle|\mathbf{x}^{(n)}\right]=\mathbf{k}_n\mathrm{E}\left[e^{-j\theta_n}\middle|\mathbf{x}^{(n)}\right]. \quad (22)$$

Then,

$$\mathrm{MCE}\left(\hat{\theta}_n\right)\geq 2-2\left(\mathbf{k}_n^H\mathbf{B}_n^{-1}\mathbf{k}_n\right)^{-\frac{1}{2}} \quad (23)$$

for any estimator $\hat{\theta}_n$ of θ_n.

Proof 1: The proof follows the lines of the proof in [24] with extension to vector parameter estimation. ■

The proof is based on the covariance inequality (e.g. [4, p. 33]), where Condition C.2 ensures that the resulting bound is independent of any specific estimator of the state parameter θ_n. It can be seen that the lower bound in (23) is based on the expression $\mathbf{k}_n^H\mathbf{B}_n^{-1}\mathbf{k}_n$, that results from the covariance inequality. This expression is transformed in a nonlinear fashion in order to bound the nonlinear MCE risk. Under the assumption that the subset of $\Omega_{\mathbf{x}^{(n)}}$ in which $\mathrm{E}\left[e^{-j\theta_n}\middle|\mathbf{x}^{(n)}\right]\neq 0$ is not empty, the vector $\mathbf{k}_n$ can be obtained from (22) in which $\mathbf{x}^{(n)}$ is substituted by any arbitrary observation vector $\mathbf{x}_0^{(n)}\in\Omega_{\mathbf{x}^{(n)}}$ with $\mathrm{E}\left[e^{-j\theta_n}\middle|\mathbf{x}_0^{(n)}\right]\neq 0$.

In the following subsection, the cyclic BCRB for periodic stochastic filtering is derived from the general bound in (23), based on a certain choice of the auxiliary function $\mathbf{v}_n$. In addition, an efficient recursive method for the cyclic BCRB computation is proposed, based on the recursive algorithm in [11].

B. The cyclic BCRB for periodic stochastic filtering

Let

$$\mathbf{J}_n^{(\mathrm{p})}\triangleq\mathrm{E}\left[\frac{\partial^T\log f_n^{(\mathrm{p})}\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)}{\partial\boldsymbol{\theta}^{(n)}}\frac{\partial\log f_n^{(\mathrm{p})}\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)}{\partial\boldsymbol{\theta}^{(n)}}\right] \quad (24)$$

denote the nth step periodic Bayesian Fisher information matrix (PBFIM). Under mild assumptions, it can be verified that the (m,q)th element of $\mathbf{J}_n^{(\mathrm{p})}$ is equal to $-\mathrm{E}\left[\frac{\partial^2\log f_n^{(\mathrm{p})}\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)}{\partial\theta_m\partial\theta_q}\right]$. We assume that the following regularity conditions are satisfied:

C.3) $f_n^{(\mathrm{p})}$ is absolutely continuous w.r.t. $\boldsymbol{\theta}^{(n)}\in\Omega_{\boldsymbol{\theta}^{(n)}},\ \forall\mathbf{x}^{(n)}\in\Omega_{\mathbf{x}^{(n)}}$.

C.4) The nth step PBFIM, $\mathbf{J}_n^{(\mathrm{p})}$, is a nonsingular matrix.

By substituting the auxiliary function

$$\mathbf{v}_{\mathrm{BCRB},n}\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)\triangleq\begin{cases}\left[1,\frac{\partial\log f_n^{(\mathrm{p})}\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)}{\partial\boldsymbol{\theta}^{(n)}}\right]^T, & f_n^{(\mathrm{p})}\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)>0\\ \left[1,\mathbf{0}_{n+1}^T\right]^T, & \text{otherwise}\end{cases}, \quad (25)$$

in (21) and (22), we obtain

$$\mathbf{B}_n=\begin{bmatrix}1 & \mathbf{0}_{n+1}^T\\ \mathbf{0}_{n+1} & \mathbf{J}_n^{(\mathrm{p})}\end{bmatrix} \quad (26)$$

and

$$\mathbf{k}_n=\left[1,\mathbf{0}_n^T,j\right]^T, \quad (27)$$

respectively. The off-diagonal blocks in the block diagonal matrix $\mathbf{B}_n$ are zero, since from Condition C.3,

$$\mathrm{E}\left[\frac{\partial\log f_n^{(\mathrm{p})}\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)}{\partial\boldsymbol{\theta}^{(n)}}\right]=\mathbf{0}_{n+1}^T.$$

It can also be shown that

$$\mathrm{E}\left[e^{-j\theta_n}\frac{\partial\log f_n^{(\mathrm{p})}\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)}{\partial\theta_i}\middle|\mathbf{x}^{(n)}\right]=\begin{cases}0, & i=0,\ldots,n-1\\ j\mathrm{E}\left[e^{-j\theta_n}\middle|\mathbf{x}^{(n)}\right], & i=n\end{cases},$$

which elucidates the result in (27). By inserting (26) and (27) in (23), one obtains the following nth step cyclic BCRB on the MCE of $\hat{\theta}_n$:

$$\text{MCE}\left(\hat{\theta}_n\right) \geq C_{\text{BCRB},n} \triangleq 2 - 2\left(1 + \left[\left(\mathbf{J}_n^{(\text{p})}\right)^{-1}\right]_{n+1,n+1}\right)^{-\frac{1}{2}}. \tag{28}$$

The MCE bound in (28) utilizes information regarding the current and previous states, since it is based on the complete PBFIM, $\mathbf{J}_n^{(\text{p})}$. It can be seen that the nth step PBFIM in (24) is obtained from the conventional nth step Bayesian Fisher information matrix (BFIM) (see e.g. [11]), $\mathbf{J}_n \triangleq \text{E}\left[\frac{\partial^T \log f_n\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)}{\partial \boldsymbol{\theta}^{(n)}} \frac{\partial \log f_n\left(\mathbf{x}^{(n)},\boldsymbol{\theta}^{(n)}\right)}{\partial \boldsymbol{\theta}^{(n)}}\right]$, by replacing the joint pdf f_n with its periodic extension, $f_n^{(\text{p})}$. As a result, Condition C.3 requires absolute continuity of the function $f_n^{(\text{p})}$, while the nth step BCRB requires absolute continuity of f_n w.r.t. $\boldsymbol{\theta}^{(n)}$, $\forall \mathbf{x}^{(n)} \in \Omega_{\mathbf{x}^{(n)}}$. Since $f_n^{(\text{p})}$ is smoother than f_n, the regularity conditions of the nth step cyclic BCRB are less restrictive than those of the conventional nth step BCRB. For example, for uniform prior distribution of θ_0 the conventional regularity conditions are not satisfied and the nth step BCRB does not exist (see e.g. [39]), while the nth step cyclic BCRB may exist, as presented in the example in Section V.

It can be seen that at each time step n, the lower bound in (28) requires the inversion of $\mathbf{J}_n^{(\text{p})}$. This task can be very difficult for large n since the size of $\mathbf{J}_n^{(\text{p})}$ grows linearly with n. Therefore, we are interested to find a recursive solution that does not involve the inversion of $\mathbf{J}_n^{(\text{p})}$.

In the following, we propose an algorithm for computing $\left[\left(\mathbf{J}_n^{(\text{p})}\right)^{-1}\right]_{n+1,n+1}$ without manipulating a linearly growing matrix. This algorithm follows the lines of [11] for the conventional nth step BCRB. We define

$$\xi_n^{(\text{p})} \triangleq \frac{1}{\left[\left(\mathbf{J}_n^{(\text{p})}\right)^{-1}\right]_{n+1,n+1}}, \tag{29}$$

the equivalent periodic Bayesian Fisher information for the estimation of θ_n, which reflects the effect of other unknown parameters. By substituting (29) in (28) the nth step cyclic BCRB for the estimation of θ_n can be written as

$$\text{MCE}\left(\hat{\theta}_n\right) \geq C_{\text{BCRB},n} = 2 - 2\left(1 + \frac{1}{\xi_n^{(\text{p})}}\right)^{-\frac{1}{2}}. \tag{30}$$

We wish to obtain a recursive computation of $\xi_n^{(\text{p})}$ that does not involve the inversion of the matrix $\mathbf{J}_n^{(\text{p})}$. By using (7), the following recursive relation between $f_{i+1}^{(\text{p})}$ and $f_i^{(\text{p})}$ can be derived:

$$\begin{aligned} & f_1^{(\text{p})}\left(\boldsymbol{\beta}^{(1)}, \boldsymbol{\alpha}^{(1)}\right) = \\ & f_{\theta_0}^{(\text{p})}(\alpha_0)\, f_{\theta_1|\theta_0}^{(\text{p})}(\alpha_1|\alpha_0)\, f_{\mathbf{x}_1|\theta_1}^{(\text{p})}(\boldsymbol{\beta}_1|\alpha_1) \end{aligned} \tag{31}$$

and

$$\begin{aligned} & f_{i+1}^{(\text{p})}\left(\boldsymbol{\beta}^{(i+1)}, \boldsymbol{\alpha}^{(i+1)}\right) = \\ & f_i^{(\text{p})}\left(\boldsymbol{\beta}^{(i)}, \boldsymbol{\alpha}^{(i)}\right) f_{\theta_{i+1}|\theta_i}^{(\text{p})}(\alpha_{i+1}|\alpha_i) \\ & f_{\mathbf{x}_{i+1}|\theta_{i+1}}^{(\text{p})}(\boldsymbol{\beta}_{i+1}|\alpha_{i+1}),\ i = 1, 2, \ldots. \end{aligned} \tag{32}$$

Proposition 2: The sequence $\left\{\xi_i^{(\text{p})}\right\}_{i=0}^{\infty}$ obeys the following recursion

$$\xi_{i+1}^{(\text{p})} = D_{i,(2,2)}^{(\text{p})} - \frac{\left(D_{i,(1,2)}^{(\text{p})}\right)^2}{\xi_i^{(\text{p})} + D_{i,(1,1)}^{(\text{p})}}, \tag{33}$$

where

$$D_{i,(1,1)}^{(\text{p})} \triangleq -\text{E}\left[\frac{\partial^2 \log f_{\theta_{i+1}|\theta_i}^{(\text{p})}(\theta_{i+1}|\theta_i)}{\partial \theta_i^2}\right], \tag{34}$$

$$D_{i,(1,2)}^{(\text{p})} \triangleq -\text{E}\left[\frac{\partial^2 \log f_{\theta_{i+1}|\theta_i}^{(\text{p})}(\theta_{i+1}|\theta_i)}{\partial \theta_i \partial \theta_{i+1}}\right], \tag{35}$$

and

$$\begin{aligned} D_{i,(2,2)}^{(\text{p})} \triangleq & -\text{E}\left[\frac{\partial^2 \log f_{\theta_{i+1}|\theta_i}^{(\text{p})}(\theta_{i+1}|\theta_i)}{\partial \theta_{i+1}^2}\right] \\ & -\text{E}\left[\frac{\partial^2 \log f_{\mathbf{x}_{i+1}|\theta_{i+1}}^{(\text{p})}(\mathbf{x}_{i+1}|\theta_{i+1})}{\partial \theta_{i+1}^2}\right], \end{aligned} \tag{36}$$

$\forall i = 0, 1, 2, \ldots$, where the algorithm is initialized with

$$\xi_0^{(\text{p})} = -\text{E}\left[\frac{\text{d}^2 \log f_{\theta_0}^{(\text{p})}(\theta_0)}{\text{d}\theta_0^2}\right] \tag{37}$$

Proof 2: The proof is similar to the proof in [11], for the nth step BCRB computation, with the pdf periodic extensions taking the role of the conventional pdfs. ■

V. Example - Phase Tracking

In this section, the proposed cyclic BCRB is applied for phase tracking of a sinusoid embedded in noise:

$$x_i = Ae^{j\theta_i} + \nu_i,\ i = 1, 2, \ldots, \tag{38}$$

where $\{\nu_i\}$ is an independent identically distributed (i.i.d.) complex circularly symmetric zero mean Gaussian noise with known variance σ^2 and $A > 0$ is a known amplitude. The state model is given by

$$\theta_{i+1} = [\theta_i + w_i]_{2\pi},\ i = 0, 1, 2, \ldots, \tag{39}$$

where $\{w_i\}$ is an i.i.d. von Mises noise with known circular mean μ and concentration κ, i.e.

$$f_{w_i}(\zeta) = \begin{cases} \frac{e^{\kappa\cos(\zeta-\mu)}}{2\pi I_0(\kappa)}, & \zeta \in [-\pi, \pi) \\ 0, & \text{otherwise} \end{cases}, \quad i = 0, 1, 2, \ldots,$$

where I_m is the modified Bessel function of order m. It is assumed that the sequences $\{w_i\}$ and $\{\nu_i\}$ are statistically independent as well as independent of past and present states. In the following, it is assumed that $\mu = 0$. The von Mises distribution is one of the most popular distributions for modeling random parameters with periodic nature and is analogous to the Gaussian distribution on the real axis (see e.g. [29], [35], [38], [43]), where the parameters μ and $\frac{1}{\kappa}$ are analogous to the corresponding mean and variance, respectively [38, p. 41]. In addition, it is assumed that the prior distribution of θ_0 is uniform, i.e. $\theta_0 \sim U(-\pi, \pi)$. For this case, the pdf periodic extensions from (4), (5), and (6) are given by

$$f_{\theta_0}^{(\mathrm{p})}(\alpha_0) = \frac{1}{2\pi}, \ \forall \alpha_0 \in \mathbb{R}, \tag{40}$$

$$f_{\theta_i|\theta_{i-1}}^{(\mathrm{p})}(\alpha_i|\alpha_{i-1}) = \frac{e^{\kappa\cos(\alpha_i-\alpha_{i-1})}}{2\pi I_0(\kappa)}, \tag{41}$$

$\forall \alpha_i, \alpha_{i-1} \in \mathbb{R}, \ i = 1, 2, \ldots$, and

$$f_{x_i|\theta_i}^{(\mathrm{p})}(\beta_i|\alpha_i) = \frac{e^{-\frac{\left|\beta_i - Ae^{j\alpha_i}\right|^2}{\sigma^2}}}{\pi\sigma^2}, \tag{42}$$

$\forall \alpha_i \in \mathbb{R}, \ \beta_i \in \mathbb{C}, \ i = 1, 2, \ldots$, respectively. It can be shown that by substituting (40)-(42) in (34)-(37), one obtains

$$D_{i,(1,1)}^{(\mathrm{p})} = \frac{\kappa I_1(\kappa)}{I_0(\kappa)}, \tag{43}$$

$$D_{i,(1,2)}^{(\mathrm{p})} = -\frac{\kappa I_1(\kappa)}{I_0(\kappa)}, \tag{44}$$

$$D_{i,(2,2)}^{(\mathrm{p})} = \frac{\kappa I_1(\kappa)}{I_0(\kappa)} + 2\mathrm{SNR}, \tag{45}$$

and

$$\xi_0^{(\mathrm{p})} = 0, \tag{46}$$

respectively, $\forall i = 0, 1, 2, \ldots$, where the signal-to-noise ratio (SNR) is defined as $\mathrm{SNR} \triangleq \frac{A^2}{\sigma^2}$. By substituting (43)-(45) in the recursive formula from (33) and initializing it with (46), one obtains the equivalent periodic Bayesian Fisher information and consequently the cyclic BCRB from (30) for the state estimation at each time step $n = 1, 2, \ldots$. It should be noted that the conventional BCRB for stochastic filtering, from [11], does not exist for this case due to regularity assumptions.

In the following, the proposed cyclic BCRB is evaluated and compared with the MCEs of the KGH filter [37], denoted as $\hat{\theta}_{\mathrm{KGH},n}^{(D)}$, with $D = 3$ deterministic samples, the WL filter [25], [31], denoted as $\hat{\theta}_{\mathrm{WL},n}^{(L)}$, with $L = 5$ Fourier coefficients, and the particle filter [8], denoted as $\hat{\theta}_{\mathrm{PAR},n}^{(S)}$, with $S = 20$ particles, where n is the time step. The MCEs of these filters are evaluated using 10,000 Monte-Carlo trials and presented in Fig. 1 versus the time step n for $\kappa = 100$ and $\sigma^2 = 1$. It can be seen that the cyclic BCRB closely predicts the performance of the KGH filter that outperforms the WL and particle filters. In Fig. 2, the MCEs of the filters are presented versus the state noise concentration, κ, for $\sigma^2 = 1$ at $n = 12$ time step. It can be seen that for $\kappa > 100$, the cyclic BCRB closely predicts the performance of the KGH filter. In Fig. 3, the MCEs of the filters are presented versus SNR for $\kappa = 100$ after $n = 12$ time steps. It can be seen that for SNR greater than 8 dB, the cyclic BCRB is achieved by all the filters.

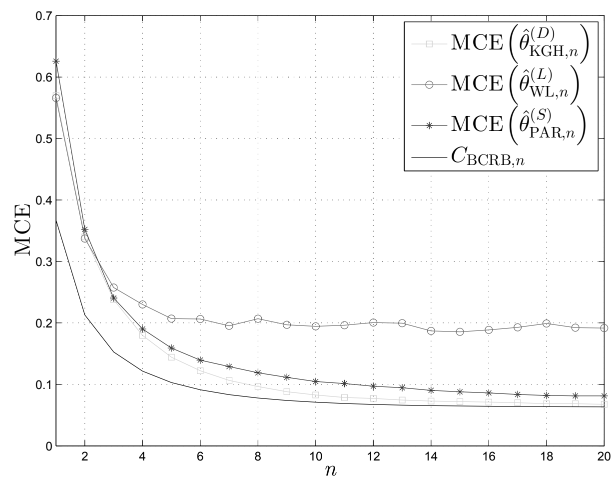

Fig. 1. The MCEs of KGH filter with $D = 3$ deterministic samples, WL filter with $L = 5$ Fourier coefficients, and particle filter with $S = 20$ particles and $C_{\mathrm{BCRB},n}$ versus the time step n.

VI. Conclusion

In this paper, we propose a Cramér-Rao-type MCE lower bound, denoted as the cyclic BCRB, which is appropriate for periodic stochastic filtering problems. It is shown that the cyclic BCRB requires less restrictive regularity conditions than the conventional BCRB. The performance of the proposed bound is compared to the MCEs of the KGH, WL, and particle filters for phase tracking problem with additive Gaussian measurement noise and von Mises distributed state noise. It is shown that the cyclic BCRB is valid for any periodic stochastic filter and that it can be achieved by existing filters.

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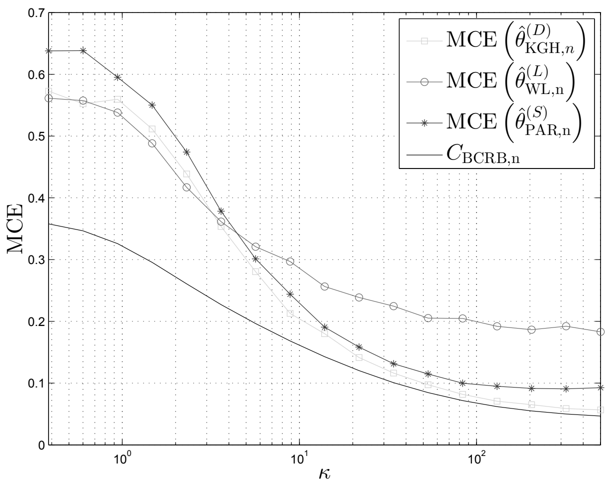

Fig. 2. The MCEs of KGH filter with $D = 3$ deterministic samples, WL filter with $L = 5$ Fourier coefficients, and particle filter with $S = 20$ particles and $C_{\mathrm{BCRB},n}$ versus the state noise concentration, κ, at $n = 12$ time step.

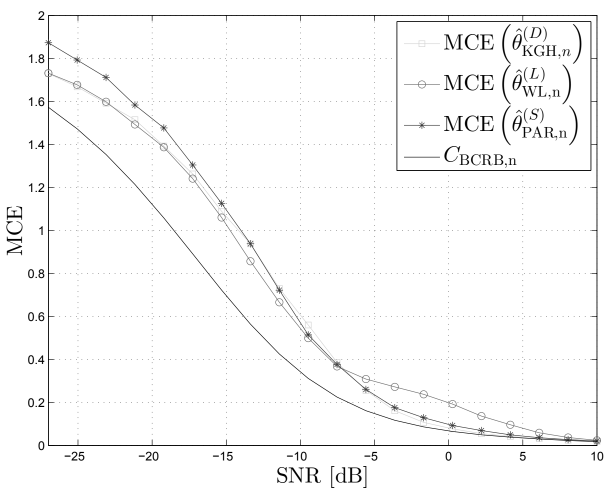

Fig. 3. The MCEs of KGH filter with $D = 3$ deterministic samples, WL filter with $L = 5$ Fourier coefficients, and particle filter with $S = 20$ particles and $C_{\mathrm{BCRB},n}$ versus SNR at $n = 12$ time step.

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