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Bernoulli Filter with Linear Equality Constraints

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Abstract - Bernoulli filters are used to estimate the states of dynamic systems recently. However, in the application of tracking ground moving targets in clutter, some valuable information tend to be ignored, such as the road constraints. The road constraints, as prior information about the state, should be incorporated into the dynamic modeling process of the filtering algorithm. In this paper, a method of incorporating state linear equality constraints (LEC) into the Bernoulli filter called the LEC Bernoulli filter (LECBF) is presented for tracking ground moving target. In this case, the Bernoulli filter is improved by exploiting road constraints, and can achieve a better filtering performance than the original unconstrained Bernoulli filter. Finally, an illustrative example is provided to show the effectiveness and efficiency of the LECBF.

Keywords: Bernoulli filters; ground target tracking; road constraints; dynamics modeling; linear equality constraint

1 Introduction

Control with constraints is useful in some engineering applications such as target tracking [1], fault diagnosis [2], robotics [3], navigation [4] and others [5]. It is increasingly applied in the tracking of ground moving targets. The state estimations of many dynamic systems are required to satisfy certain constraints. There are two kinds of constraints: hard constraint and soft constraint. Hard constraint is an exact relationship between state variables which must be satisfied. Soft constraint only can be satisfied approximately or probabilistically [8]. Constraints can also be classified to equality or inequality and linear or nonlinear [5], etc. Researchers have made great efforts to estimate constrained states and proposed several equality or inequality constrained estimation methods [5] [6] [7] [8] including model reduction [9] [10], perfect measurements [11] [12] and estimation projection [7] [13]. Furthermore, [18] incorporates digital road maps for road constrained targets, which leads to more precise tracks.

Bernoulli filter is an optimal Bayes filter also known as joint target detection and tracking filter (JoTT) [14] [15]. In the application of Bernoulli filter, there is some known valuable information, such as geographic constraint that is often neglected because they do not fit easily into the structure of the Bernoulli filter.

The problem of estimating target states which constrained by a class of hard linear equality is considered in this paper. The road constraint is a type of prior information and it should be incorporated into the dynamic modeling. A method of oblique projection is utilized to decompose the original unconstrained state into two mutually uncorrelated terms [8] and makes it fit easily into the structure of the Bernoulli filter.

Section 2 presents a brief summary of the Bernoulli filter solution without consideration of any state constraints. Section 3 formulates the hard LEC problem and presents the algorithm of LECBF. Section 4 shows some simulation results, and finally described some concluding remarks and suggestions for further work.

2 Bernoulli Filter

The discrete-time formulation of dynamic filtering problem in the Bayesian framework is as follows [16]. Suppose the state vector $x_k \in \chi$ provides the complete specification of the state of a dynamic system at time k. Here $\chi \subseteq \mathbb{R}^{n_x}$ is the state space, while k is the discrete-time index. The dynamic system is described by the following two equations:

$$x_k = Fx_{k-1} + v_{k-1} \tag{1}$$

$$z_k = Hx_k + w_k \tag{2}$$

where F is the state transition matrix and H is the measurement matrix. v_{k-1} and w_k are the zero-mean process and measurement noise respectively.

In tracking or filtering we are interested in the probability density of the state x_k at time k given all measurements $z_{1:k} = (z_1, ..., z_k)$ up to time k, denoted by $p_k(x_k \mid z_{1:k})$.

Assuming the initial posterior density $p(x_0)$ is known. A two-step procedure can be used to recursively estimate the posterior density of the state at time k :

$$p_{k|k-1}(x_k \mid z_{1:k-1}) = \int f_{k|k-1}(x_k \mid x) p_{k-1}(x \mid z_{1:k-1}) dx \quad (3)$$

$$p_k(x_k \mid z_{1:k}) = \frac{g_k(z_k \mid x_k) p_{k|k-1}(x_k \mid z_{1:k-1})}{\int g_k(z_k \mid x) p_{k|k-1}(x \mid z_{1:k-1}) dx} \quad (4)$$

where $f_{k|k-1}(x_k \mid x_{k-1})$ is the transitional density and $g_k(z_k \mid x_k)$ is the likelihood function.

2.1 Bernoulli RFS

A random finite set (RFS) is a random variable that takes values as unordered finite sets [14]. The target state X is modeled as a Bernoulli RFS. A Bernoulli RFS on χ has probability $1-q$ of being empty, and probability q of being a singleton whose only element is distributed according to a probability density $s(x)$ .The probability density of a Bernoulli RFS is given by:

$$f(X) = \begin{cases} 1-q, & X = \varnothing \\ q \cdot s(x), & X = \{x\} \\ 0, & otherwise \end{cases} \quad (5)$$

2.2 Bernoulli Filter

The Bernoulli filter, as a sequential Bayesian estimator, recursively estimates the posterior density of object state through the prediction and update stages by using the dynamic model and measurement model. The posterior density at time k can be completely specified by two quantities: the posterior probability of object existence $q_{k|k}$ and the posterior density of the state $s_{k|k}(x)$.

Reference [14] and [17] have originally derived the prediction and update equations of the Bernoulli filter. At time $k-1$, let the posterior probability of object existence be $q_{k-1|k-1}$, the posterior density of the kinematic state of the target be $s_{k-1|k-1}(x)$. Now the prediction and update steps of the Bernoulli recursion are recalled [17]:

Prediction step:

If the posterior density f_{k-1} at time $k-1$ is a Bernoulli of the form $f_{k-1} = \{p_{k-1}, s_{k-1}(x)\}$, then the predicted density $f_{k|k-1}$ to time k is also a Bernoulli and is given by $f_{k|k-1} = \{p_{k|k-1}, s_{k|k-1}(x)\}$. The prediction equations for the probability of existence and posterior density of the state are given by:

$$q_{k|k-1} = p_b(1-q_{k-1|k-1}) + p_s q_{k-1|k-1} \quad (6)$$

$$s_{k|k-1}(x) = \frac{p_b(1-q_{k-1|k-1}) b_{k|k-1}(x)}{q_{k|k-1}} + \frac{p_s q_{k-1|k-1} \int f_{k|k-1}(x \mid x') s_{k-1|k-1}(x') dx'}{q_{k|k-1}} \quad (7)$$

where, p_b is the probability of object "birth" during the sampling interval, and p_s is the probability of target "survival" during the sampling interval. If the object appears during the sampling interval, $b_{k|k-1}(x)$ denotes its birth density.

Update Step:

If the predicted density $f_{k|k-1}$ to time k is a Bernoulli of the form $f_{k|k-1} = \{p_{k|k-1}, s_{k|k-1}(x)\}$, then the updated density $f_{k|k}$ at time k is also a Bernoulli and is given by $f_{k|k} = \{p_{k|k}, s_{k|k}(x)\}$ where

$$q_{k|k} = \frac{1-\Delta_k}{1-q_{k|k-1}\Delta_k} q_{k|k-1} \quad (8)$$

$$s_{k|k}(x) = \frac{1-p_d(x) + p_d(x)\sum_{z \in \mathbf{Z}_k} \frac{g_k(z \mid x)}{\lambda c(z)}}{1-\Delta_k} s_{k|k-1}(x) \quad (9)$$

$$\Delta_k = \int p_d(x) s_{k|k-1}(x) dx - \sum_{z \in \mathbf{Z}_k} \frac{\int p_d(x) g_k(z \mid x) s_{k|k-1}(x) dx}{\lambda c(z)}$$

(10)

If $p_d = const$ (independent of the target state), then (10) simplifies to :

$$\Delta_k = p_d\Big(1-\sum_{z \in \mathbf{Z}_k} \frac{\int g_k(z \mid x) s_{k|k-1}(x) dx}{\lambda c(z)}\Big) \quad (11)$$

where p_d is the probability of detecting the target. $g_k(z \mid x)$ is the conventional likelihood function of the target-generated measurement z . $c(z)$ and λ are the PDF and average number of clutter respectively.

3 Linear Equality Constrained Bernoulli Filter

The incorporation of prior information such as road constraints can be used to improve the accuracy of tracking algorithm. In this section the linear equality constrained Bernoulli filter is presented by utilizing the Gaussian mixture Bernoulli filter to accommodate road constraints.

Consider a dynamic system whose state satisfies the hard LEC:

$$C_k x_k = d_k, k = 0,1,2,... \qquad (12)$$

where k is the time step, x_k is the state, C_k is a known matrix and d_k is a known vector but may be time varying. Based on the LEC (12), define a simple random vector $r \in \mathbb{R}^m$ as

$$r = Cx^u \qquad (13)$$

the superscript u implies unconstrained state. Let $\Sigma_r = \mathrm{cov}(r)$, then an oblique (not orthogonal) projector P is used to decompose the state of the original unconstrained state x^u into two mutually uncorrelated terms [8]:

$$x^u = Px^u + (I-P)Ar \qquad (14)$$

$$\Sigma^u = P\Sigma^u P^T + (I-P)A\Sigma_r A^T (I-P)^T \qquad (15)$$

where $P = I - \Sigma^u C^T (C\Sigma^u C^T)^{-1} C$, $A = C^T (CC^T)^{-1}$

The constrained state estimation of linear systems are considered in this paper. The original unconstrained dynamic model and measurement model are:

$$x_k^u = F_{k-1} x_{k-1}^u + w_{k-1}^u \qquad (16)$$

$$z_k = H_k x_k + v_k \qquad (17)$$

where F_{k-1} and H_k are the state transition and measurement matrices. w_{k-1}^u and v_k are the zero-mean process and measurement noise with covariance Q^u and R respectively. Under the following assumptions:

1) Each target evolves and generates measurements independently.
2) Target births follow a Bernoulli RFS independent of target survivals.
3) Clutters, which are not too dense, and independent of target-generated measurements, follow Poisson distribution.
4) The dynamic and measurement model are both linear Gaussian forms.

$$f_{k|k-1}(x \mid \zeta) = N(x; F_{k-1}\zeta, Q_{k-1}) \qquad (18)$$

$$g_k(z \mid x) = N(z; H_k x, R_k) \qquad (19)$$

5) The survival and detection probability are state independent, i.e.

$$p_{S,k}(x) = p_{S,k}, p_{d,k}(x) = p_{d,k} \qquad (20)$$

6) The density of the birth is linear Gaussian mixtures of the form

$$b_{k|k-1}(\mathrm{x}) = \sum_{i=1}^{N_{b,k}} w_{b,k}^{(i)} \mathrm{N}(\mathrm{x}; m_{b,k}^{(i)}, Q_{b,k}^{(i)}) \qquad (21)$$

where $w_{b,k}^{(i)}$, $N_{b,k}$, $m_{b,k}^{(i)}$, $Q_{b,k}^{(i)}$ are given model parameters that determine the shape of the birth density.

Linear equality constrained state estimation for the original unconstrained system model can be performed by using the Bernoulli filter . A complete algorithm of the LECBF is summarized as following:

Table 1 LECBF Algorthim

1. Parameter Calculation: $(k = 0,1,2...)$

matrix $A_k = C_k^T (C_k C_k^T)^{-1}$

projectors

$$P_0 = I - \Sigma_0^u C_0^T (C_0 \Sigma_0^u C_0^T)^{-1} C_0$$

$$P_k = I - Q_{k-1}^u C_k^T (C_k Q_{k-1}^u C_k^T)^{-1} C_k \text{ for } k \ge 1$$

parameters: $Q_{k-1} = P_k Q_{k-1}^u P_k^T$, $d_k^* = (I - P_k) A_k d_k$

2. Initialization：

$$s_0(x) = \sum_{i=1}^{N_0} w_0^{(i)} \mathrm{N}(x; \hat{m}_0^{(i)}, \hat{\Sigma}_0^{(i)})$$

where , $\sum_{i=1}^{N_0} w_0^{(i)} = 1$, $\hat{m}_0 = P_0 m_0^{(i)} + d_0^*$, $\hat{\Sigma}_0^{(i)} = P_0 \Sigma_0^{(i)}$

3. Coefficient Resetting:

$$F_{k-1}^* = P_k F_{k-1}, G_{k-1}^* = P_k G_{k-1}$$

4. Filtering ($k \ge 1$)

Prediction:

$$q_{k|k-1} = p_b(1 - q_{k-1|k-1}) + p_s q_{k-1|k-1}$$

$$s_{k|k-1}(x) = \frac{p_b(1 - q_{k-1|k-1})}{q_{k|k-1}} b_{k|k-1}(x) + \frac{p_s q_{k-1|k-1}}{q_{k|k-1}} \sum_{i=1}^{N_{k-1}} w_{k-1}^{(i)} \mathrm{N}(x; \hat{m}_{k|k-1}^{(i)}, \hat{\Sigma}_{k|k-1}^{(i)})$$

with

$$\hat{m}_{k|k-1}^{(i)} = F_{k-1}^* \hat{m}_{k-1}^{(i)} + d_k^*$$

$$\hat{\Sigma}_{k|k-1}^{(i)} = Q_{k-1} + F_{k-1}^* \hat{\Sigma}_{k-1}^{(i)} (F_{k-1}^*)^T$$

Update:

$$q_{k|k} = \frac{1 - \Delta_k}{1 - q_{k|k-1}\Delta_k} q_{k|k-1}$$

$$s_{k|k}(x) = \frac{1 - p_d}{1 - \Delta_k} s_{k|k-1}(x) + \frac{p_d}{1 - \Delta_k} \sum_{z \in Z_k} \sum_{i=1}^{N_{k|k-1}} \frac{w_{k|k-1}^{(i)} q_k^{(i)}(\mathrm{z})}{\lambda c(\mathrm{z})} \mathrm{N}(x; \hat{m}_{k|k}^{(i)}, \hat{\Sigma}_{k|k}^{(i)})$$

$$\Delta_k = p_d [1 - \sum_{z \in Z_k} \sum_{i=1}^{N_{k|k-1}} \frac{w_{k|k-1}^{(i)} q_k^{(i)}(\mathrm{z})}{\lambda c(\mathrm{z})}]$$

where,

$$q_k^{(i)}(z) = N(z;\eta_{k|k-1}^{(i)}, S_{k|k-1}^{(i)})$$
$$\eta_{k|k-1}^{(i)} = H_k \hat{m}_{k|k-1}^{(i)}$$
$$S_{k|k-1}^{(i)} = H_k \hat{\Sigma}_{k|k-1}^{(i)} H_k^T + R_k$$
$$K_k^{(i)} = \hat{\Sigma}_{k|k-1}^{(i)} H_k^T [S_{k|k-1}^{(i)}]^{-1}$$
$$\hat{m}_{k|k}^{(i)}(z) = \hat{m}_{k|k-1}^{(i)} + K_k^{(i)}(z - \eta_{k|k-1}^{(i)})$$
$$\hat{\Sigma}_{k|k}^{(i)} = \hat{\Sigma}_{k|k-1}^{(i)} - K_k^{(i)} S_{k|k-1}^{(i)} (K_k^{(i)})^T$$

5. Pruning and merging Gaussian components:

given $\left\{ w_k^{(i)}, m_k^{(i)}, \Sigma_k^{(i)} \right\}_{i=1}^{N_k}$, a truncation threshold T, a merging threshold U, and a maximum allowable number of Gaussian terms $N_{\max}$.

Set $l = 0$, and $I = \{ \ i = 1,...,N \,|\, w_k^{(i)} > T \ \}$

repeat

$$l = l + 1$$
$$j = \arg\max_{i \in I} w_k^{(i)}$$
$$L = \left\{ i \in I \,|\, \left(m_k^{(i)} - m_k^{(j)}\right)^T \left(\Sigma_k^{(i)}\right)^{-1} \left(m_k^{(i)} - m_k^{(j)}\right) \le U \right\}$$
$$\hat{w}_k^{(l)} = \sum_{i \in L} w_k^{(i)}$$
$$\hat{m}_k^{(l)} = \frac{1}{\hat{w}_k^{(l)}} \sum_{i \in L} w_k^{(i)} m_k^{(i)}$$
$$\hat{\Sigma}_k^{(l)} = \frac{1}{\hat{w}_k^{(l)}} \sum_{i \in L} w_k^{(i)} (\Sigma_k^{(i)} + (\hat{m}_k^{(l)} - m_k^{(i)})(\hat{m}_k^{(l)} - m_k^{(i)})^T)$$
$$I = I \setminus L$$

Until

$$I = \phi$$

if $l > N_{\max}$ then replace $\{\hat{w}_k^{(i)}, \hat{m}_k^{(i)}, \hat{\Sigma}_k^{(i)}\}_{i=1}^{l}$ by those of the $N_{\max}$ Gaussians with largest weights.

output $\{\hat{w}_k^{(i)}, \hat{m}_k^{(i)}, \hat{\Sigma}_k^{(i)}\}_{i=1}^{l}$ as the pruned and merged approximation to the posterior density.

4 Simulation Analysis

In this section, a simple example is presented to show the performance comparison between the LECBF and BF, we consider an example of tracking a Land-Based vehicle (adopted in [7] first and then in [6] and [8]). The vehicle dynamics and measurements can be approximated by the following equations:

$$x_{k+1} = F x_k + w_k \tag{22}$$
$$z_k = H x_k + v_k \tag{23}$$

with

$$F = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$G = \begin{bmatrix} 0 \\ 0 \\ T\sin\theta_i \\ T\cos\theta_i \end{bmatrix}, i = 1, 2$$
$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{24}$$

The state vector $x = [x_1 \quad x_2 \quad \dot{x}_1 \quad \dot{x}_2]^T$ (where the first two elements (x_1, x_2) are the northerly and easterly positions, the last two elements $(\dot{x}_1, \dot{x}_2)$ are the northerly and easterly velocities), w_k and v_k are uncorrelated zero-mean white process noise and measurement noise, respectively. T is the sampling period and θ_i stands for the heading angle (measured counterclockwise from due east) of the vehicle located in road segment i . During certain times the vehicle is travelling on-road, in which case the state estimation is constrained. For instances, the vehicle moves on different road segments with known directions, starting from the initial state x_0 heading θ_1, after 30s, it changes the direction of the velocity and turns to another segment, and then performs a nearly constant velocity motion with heading θ_2 which is maintained for 30s until the end. The dynamic constraints of the vehicle in the two segments can be expressed by

$$[0 \quad 0 \quad 1 \quad -\tan\theta_i] x_k = 0, \quad i = 1, 2 \tag{25}$$

We set $T = 2s, \theta_1 = 60^\circ$ and $\theta_2 = 45^\circ$, and the covariance of the measurement noise $R = diag(400, 400, 20)$. The initial state is $x_0 = [0 \quad 0 \quad 11.8301 \quad 6.8301]^T$, and the covariance of the process noise in the two segments are

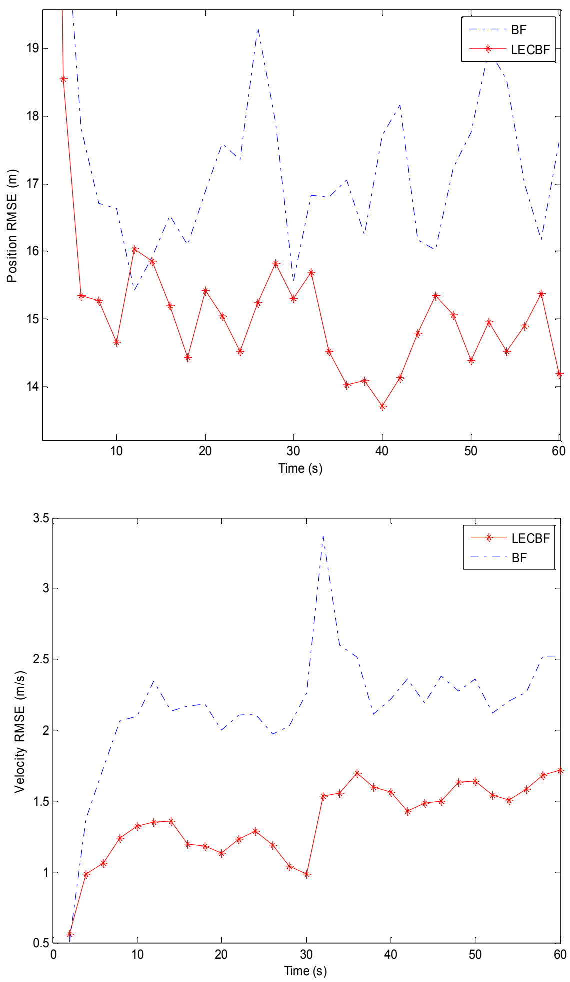

Figure 1. Estimation errors versus time for tracking a land-based vehicle

$$Q_1 = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 64 & 0 & 0 \\ 0 & 0 & 0.9474 & 0.5470 \\ 0 & 0 & 0.5470 & 0.3158 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 64 & 0 & 0 \\ 0 & 0 & 0.8571 & 0.8571 \\ 0 & 0 & 0.8571 & 0.8571 \end{bmatrix} \qquad (26)$$

The state of the vehicle evolving as (16) with $Q_i (i = 1, 2)$ automatically satisfies the constraint (19), and the observations generated by the sensor are about the constrained state.

In this simulation, we use the LECBF to estimate the position of the vehicle and compare its performance with BF, These algorithms utilize the common unconstrained model over the whole time horizon, which is identical to the system model (16)-(20) except that the progress noise covariance Q_1 and Q_2 are replaced by $Q^u = diag(16 \quad 64 \quad 1 \quad 6)$. Here Q^u is designed without much consideration of the constraints. Besides, the estimators share the same initialization: x_0 and

$$\Sigma_{0|0} = \begin{bmatrix} 400 & 0 & 0 & 0 \\ 0 & 400 & 0 & 0 \\ 0 & 0 & 7.5000 & 4.3301 \\ 0 & 0 & 4.3301 & 2.5000 \end{bmatrix},$$

Fig.1 shows the comparison of the RMS position and velocity errors of the estimators over 100 Monte Carlo runs. The dash-dot line is for the Bernoulli Filter without consideration of the constraint, the cross signs are for the proposed LECBF. From the two figures, we can observe that the error level of LECBF is much lower than that of unconstrained Bernoulli filter. The LECBF is superior to the unconstrained Bernoulli filter.

Since the RMSE plot does not capture the Bernoulli filter's ability to detect, Fig.2 shows the optimal subpattern assignment (OSPA) metric to reveal the improvement in both detection and estimation.

From figure 2, we can observe that the mean OSPA values of LECBF are much lower than that of unconstrained Bernoulli filter. It means that the inclusion of the constraints reduces the search space and leads to better detections.

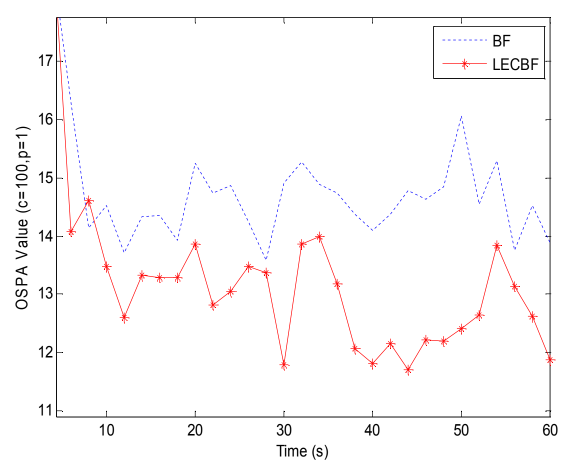

Figure 2. Mean OSPA values

The average computational loads of the two filters in terms of CPU time are shown in Table 2. It is shown that the LECBF is approximated to the unconstrained BF in cc

Table 3 Comparison of computational loads

Table 2 Comparison of computational loads

LECBF	BF
0.4147s	0.4065s

5 Conclusion

In this paper, the problem of modeling and state estimation for dynamic systems with linear equality constraints has been analyzed. As prior information, road constraints should be incorporated into the dynamic modeling. A method for incorporating linear equality constraints in the Bernoulli filter is presented in this paper. The simulation results indicate the effectiveness and efficiency of the LECBF, which is superior to that of unconstrained BF. If the state constraints are nonlinear, they can be linearized, although this may result in convergence problems. Our further work will focus on these lines.

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