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Target Motion Analysis by Inverse Triangulation

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Abstract - We propose a new method to estimate the trajectory of a source based on the two angles of the arrival lines of two waves emitted by the source but propagating with two different speeds. This difference between speeds is due to either the different natures of the waves or the different natures of the media in which the waves propagate. The source is assumed to move with a constant velocity and the observer is motionless. First we prove that such a trajectory is observable, and then we propose the maximum likelihood estimate (under Gaussian hypothesis) which is shown to be almost efficient.

Keywords: Bearings-only TMA, estimation, Fisher information matrix, Cramér-Rao lower bound, observability.

1 Introduction

When we look at a plane in the sky, most of the time we do not hear the sound of the engine coming from its direction. We have all done this experiment which allows young children to realize that sound travels slower than light. In this paper, we propose a new way to passively estimate the trajectory of a source (or target) by simultaneously exploiting the angles of the "lines of sight" (LOS) and the angles of the "lines of sound". Indeed, fusing these two pieces of information makes the trajectory of a source (whose velocity is constant) observable. Hence, the estimation can be envisaged. All these points are examined in this paper, yielding an original method. Indeed, to our knowledge, this problem has never been studied in the open literature. Still, this method is of great interest because it can be carried out in, for instance, naval contexts when a passive radar and a passive sonar are available, and more generally when a source emits two kinds of signals propagating with different speeds.

In some papers, the so-called "delayed measurements" are taken into account; for example, in [6] "out-of-sequence measurements" sent by a set of sensors and received by a single center are synchronized by estimating each time delay. In [6-12], this time delay is on-line estimated to reduce the bias of the estimator of targets position. Hence, the time delay appears as a "drawback" of the propagation phenomena. Here, it must be considered as an advantage allowing one to propose a "low-cost" estimation of the target trajectory.

Our paper is composed of three main sections:

Section 2 is devoted to the definition of our problem. We give the notations employed subsequently.

In Section 3, we prove that the trajectory of the source is observable when at least three instantaneous bearings and a delayed bearing are available.

The estimation of the trajectory is developed in Section 4. The Cramér-Rao lower bound is computed and three typical scenarios are treated to give the performance of the maximum likelihood estimator via Monte Carlo simulations. A conclusion follows.

2 Hypotheses and notations

Consider a passive motionless observer (O) and a target (T) moving with a constant velocity (or CV motion). The origin of the Cartesian coordinates of the plane in which the two protagonists are, is chosen to be at the location of the observer. In this coordinate system, the position of the target at time t is denoted $P_T(t)=[x_T(t) \quad y_T(t)]^T = R(t)[\sin\theta(t) \quad \cos\theta(t)]^T$ and its velocity is $V=[\dot{x} \quad \dot{y}]^T = v\,[\sin\gamma \quad \cos\gamma]^T$. The trajectory of the target obeys the following equation: $P_T(t)=P_T(t^*)+(t-t^*)V$, t^* being an arbitrary reference time. We denote $x_T(t^*)=x$ and $y_T(t^*)=y$. The trajectory of the source is entirely characterized by the state vector $X=[x \quad y \quad \dot{x} \quad \dot{y}]^T$. Note that whatever t is, the target and the observer are never located at the same place, *i.e.* $[x_T(t) \quad y_T(t)] \neq [0 \quad 0]$.

At any time, the observer simultaneously measures two angles: the angle of the "line of sight" (LOS) and the angle of the "line of sound". Of course, the propagation delay of the light is negligible whereas that of the sound must be taken into consideration. So, we can say that the angle of the LOS is equal to $\theta(t)$ and the angle of the line of sound $\theta_D(t)$ is delayed by $\tau(t)$: $\theta_D(t)=\theta(t-\tau(t))$. The propagation delay $\tau(t)$ satisfies the recursion $\tau(t)=\frac{R(t-\tau(t))}{c}$, where c is the propagation speed of sound in the environment. It can be computed by the following expression

$$\tau(t)=\frac{\sqrt{\left[V^T P_T(t)\right]^2+\left(c^2-v^2\right)\|P_T(t)\|^2}-V^T P_T(t)}{\left(c^2-v^2\right)},$$ proved in the Appendix.

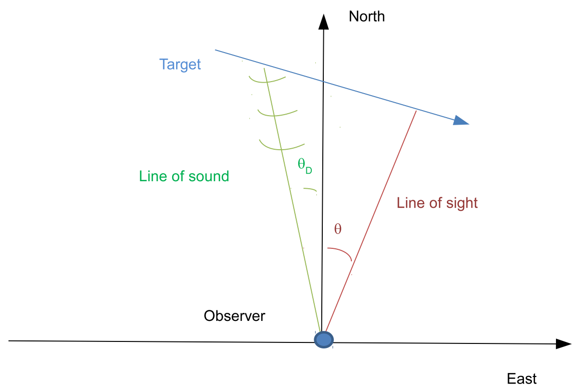

Figure 1. Example of scenario.

At time t_k, the observer acquires the measured angles $\theta_m(t_k)$ and $\theta_{D,m}(t_k)$:

$\theta_m(t_k)=\theta(t_k)+\varepsilon(t_k)$ and

$\theta_{D,m}(t_k)=\theta_D(t_k)+\varepsilon_D(t_k)$, for $k=1,2,\ldots,N$.

$\varepsilon_D(t_k)$ and $\varepsilon(t_k)$ are the additive noise, assumed to be zero-mean and Gaussian. Their covariance matrix are respectively equal to $diag(\sigma_D^2)$ and $diag(\sigma^2)$ (assumed to be known).

We will assume subsequently that $t_k=(k-1)\Delta t$, where Δt is the measurement period. With no loss of generality, we choose $t^*=0$.

We aim to estimate the trajectory of the source from the two sets $\{\theta_m(t_k),\quad k=1,2,\ldots,N\}$ and $\{\theta_{D,m}(t_k),\quad k=1,2,\ldots,N\}$.

In the following section, we prove that the trajectory is observable from $\{\theta(t_k),\quad k=1,2,\ldots,N\}$ and $\{\theta_D(t_k),\quad k=1,2,\ldots,N\}$.

3 Observability Analysis

3.1 Preamble

First, we have the following one-way condition:

$$\theta(t)=Arg\left[x_T(t),y_T(t)\right]\ \Rightarrow\ \frac{\sin\theta(t)}{\cos\theta(t)}=\frac{x_T(t)}{y_T(t)}$$
$$\Leftrightarrow\quad x_T(t)\cos\theta(t)-y_T(t)\sin\theta(t)=0. \qquad (1)$$

Replacing $x_T(t)$ by $x+t\dot{x}$ and $y_T(t)$ by $y+t\dot{y}$ in (1) leads to

$x\cos\theta(t)-y\sin\theta(t)+t\dot{x}\cos\theta(t)-t\dot{y}\sin\theta(t)=0$.

$\Leftrightarrow R(t)\left[x\cos\theta(t)-y\sin\theta(t)+t\dot{x}\cos\theta(t)-t\dot{y}\sin\theta(t)\right]=0$

$\Leftrightarrow x\,y_T(t)-y\,x_T(t)+t\,\dot{x}\,y_T(t)-t\,\dot{y}\,x_T(t)=0$

i.e., after re-introducing the state vector, $\left[y_T(t)\quad -x_T(t)\quad t\,y_T(t)\quad -t\,x_T(t)\right]X=0$.

Remark:
We emphasize the fact that (1) is not equivalent to $\theta(t)=Arg\left[x_T(t),y_T(t)\right]$ since (1) remains valid when $\theta(t)$ is replaced by $\theta(t)+\pi$.

Proposition 1
Consider a target in CV motion, detected in three LOS at times t_1, t_2, and t_3 by a motionless observer.
If the three LOS are not equal, then the set of targets in CV motion detected in the same LOS is defined by the set of the state vector $\{X',\ \text{such as}\ \exists\lambda>0,\ X'=\lambda X\}$.

Proof:
Without loss of generality, we assume that $t_1=0$.
Consider the three different azimuths $\theta(t_1),\theta(t_2),\theta(t_3)$ of the LOS. By construction, the system

$$\begin{cases} x_T(t_1)\cos\theta(t_1)-y_T(t_1)\sin\theta(t_1)=0\\ x_T(t_2)\cos\theta(t_2)-y_T(t_2)\sin\theta(t_2)=0\\ x_T(t_3)\cos\theta(t_3)-y_T(t_3)\sin\theta(t_3)=0\end{cases}$$

can be expressed in the closed form

$$\underbrace{\begin{bmatrix} y_T(t_1) & -x_T(t_1) & 0 & 0\\ y_T(t_2) & -x_T(t_2) & t_2 y_T(t_2) & -t_2 x_T(t_2)\\ y_T(t_3) & -x_T(t_3) & t_3 y_T(t_3) & -t_3 x_T(t_3)\end{bmatrix}}_{=\mathbf{M}}X=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad (2)$$

The analysis of observability is to identify any four-dimensional vectors X' which can be a solution of (2), *i.e.* $\mathbf{M}X' = 0_3$. In other words, we have to identify the null space of $\mathbf{M}$.
We are going to prove that $\text{Rank}(\mathbf{M}) = 3$. Obviously, $\text{Rank}(\mathbf{M}) \le 3$.
Compute the determinant of the submatrix composed by the last three columns:

$$
\begin{aligned}
D_1 &= \det\begin{bmatrix} -x_T(t_1) & 0 & 0 \\ -x_T(t_2) & t_2 y_T(t_2) & -t_2 x_T(t_2) \\ -x_T(t_3) & t_3 y_T(t_3) & -t_3 x_T(t_3) \end{bmatrix} \\
&= t_2 t_3 x_T(t_1)[y_T(t_2)x_T(t_3) - x_T(t_2)y_T(t_3)] \\
&= t_2 t_3 (t_2 - t_3)(\dot{y}x - \dot{x}y)x.
\end{aligned}
$$

Now, compute the determinant of the (3×3)matrix composed by the first column and the last two columns of

$$
\begin{aligned}
\mathbf{M}: D_2 &= \det\begin{bmatrix} y_T(t_1) & 0 & 0 \\ y_T(t_2) & t_2 y_T(t_2) & -t_2 x_T(t_2) \\ y_T(t_3) & t_3 y_T(t_3) & -t_3 x_T(t_3) \end{bmatrix} \\
&= -t_2 t_3 (t_2 - t_3)(\dot{y}x - \dot{x}y)y .
\end{aligned}
$$

We get $|D_1| + |D_2| = -t_2 t_3 (t_2 - t_3)\,|\dot{y}x - \dot{x}y|\,(|x| + |y|)$. Hence $|D_1| + |D_2|$ is not equal to zero unless $\dot{y}x - \dot{x}y = 0$, *i.e.* the vectors $[x \quad y]^T$ and $[\dot{x} \quad \dot{y}]^T$ are collinear. This case, corresponding to the scenario where $\theta(t_1) = \theta(t_2) = \theta(t_3)$, is discarded by assumption.
Hence, the rank of $\mathbf{M}$ is equal to 3 and, as a consequence, the dimension of its null space is equal to 1. We conclude that $\exists\, \lambda$ s.t. $X' = \lambda X$.

This condition is necessary but not sufficient. In order to respect the equality $\theta(t_i) = Arg[\lambda x_T(t_i), \lambda y(t_i)]$ for $i \in \{1, 2, 3\}$, the scalar λ must be strictly positive.
■

Remarks: (i) The above proof is widely inspired by [1] ; a version in continuous time can be found in some pioneering papers such as [3, 4]; (ii) a similar result can be found in [2] for continuous time; (iii) consequently, we are able to compute the azimuth $\theta(t)$ at any time t.

3.2 Analysis

Proposition 2

Consider a target in CV motion, detected in three LOS at times t_1, t_2, and t_3, and a line of sound at time t_4 by a motionless observer.
Then the trajectory of this target is observable provided that the three LOS are not equal.

Proof:
We know from Proposition 1 that the set of targets detected in three LOS and a line of sound is contained in the set $\Lambda = \{X' = \lambda X_T,\ for\ \lambda > 0\}$. At time t_4, we acquire the angle of the line of sound $\theta_D(t_4)$ and we compute $\theta(t_4)$, thanks to Proposition 1.
We denote A and A', the respective positions of the target and of a λ-homothetic solution at time $t_4 - \tau(t_4)$, and B and B' their respective positions at time t_4. The speed v' of the λ-homothetic solution is obviously equal to λv.
Figure 2 illustrates the trajectory of the target and one homothetic solution:

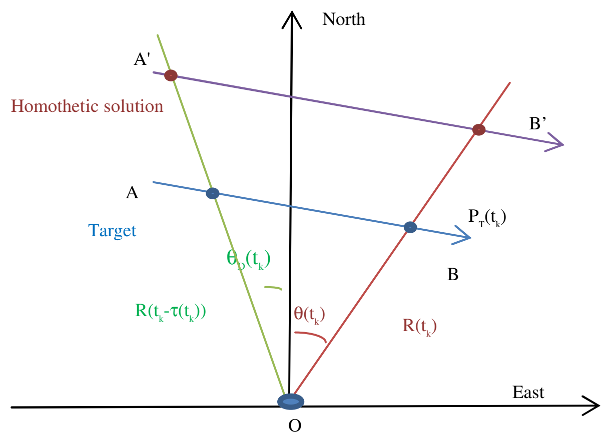

Figure 2. The target and a homothetic solution.

From the Thales' theorem (or intercept theorem), we get $\frac{A'B'}{AB} = \frac{OA'}{OA} = \lambda$
with $OA = R(t_4 - \tau(t_4)) = c\tau(t_4)$, $OA' = R'(t_4 - \tau'(t_4)) = c\tau'(t_4)$, $AB = v\,\tau(t_4)$ and $A'B' = v'\tau'(t_4)$. In these expressions, $\tau'(t_4) = \frac{R'(t_4 - \tau'(t_4))}{c}$.
Since $OA' = \lambda OA$, we get $\tau'(t_4) = \lambda\tau(t_4)$. We end up with $A'B' = \lambda^2 v\,\tau(t_4)$.
The equality $A'B' = \lambda AB$ implies hence that $\lambda^2 v\,\tau(t_4) = \lambda v\,\tau(t_4)$, so $\lambda = 1$, and X' is equal to X. •

4 Estimation

4.1 Computation of the Cramér-Rao lower bound

The Cramér-Rao lower bound (CRLB) is the inverse of the Fisher information matrix (FIM) given by the well-known formula

$$F(X)=\sum_{k=1}^{N}\frac{1}{\sigma^2}\nabla_X\theta(X,t_k)\nabla_X^T\theta(X,t_k) + \sum_{k=1}^{N}\frac{1}{\sigma_D^2}\nabla_X\theta_D(X,t_k)\nabla_X^T\theta_D(X,t_k)$$

Only the computation of $\nabla_X\theta_D(X,t_k)$ needs some development, which is presented in the Appendix.

4.2 The chosen estimator

We chose the maximum likelihood estimator (MLE) that is identical to the least squares estimator minimizing the criterion

$$C(X)=\sum_{k=1}^{N}\frac{1}{\sigma^2}[\theta_m(t_k)-\theta(X,t_k)]^2+\sum_{k=1}^{N}\frac{1}{\sigma_D^2}[\theta_{D,m}(t_k)-\theta_D(X,t_k)]^2,$$

since the noise is assumed to be Gaussian.

The Gauss-Newton routine is employed for this minimization. The Hessian is based upon the expression of the FIM presented above.

4.3 Monte Carlo simulations

In our simulations, three types of targets have been considered: a vessel (low speed), a helicopter (medium speed), and an airplane (high speed). No numerical problem has been noted. The units employed are those of the international metric system (meters for the distances and meters per second for the speeds). The Gauss-Newton routine was initialized with $X_{init}=[1000 \quad 1000 \quad 0 \quad 0]^T$ (m, m, m/s, m/s). For each example, we carried out 500 Monte-Carlo simulations. We computed the sampling standard deviation $\hat{\sigma}$ which we compared to the square root of the corresponding diagonal element of the CRLB σ_{CRLB}.

4.3.1 Vessel

We assume that the observer has a camera or a periscope (in the underwater context) and a passive sonar system. Each system measures an angle at regular instants. The target starts from $P_T(0)=[-2000 \quad 3000]^T$ (m), with a speed equal to 5 m/s and a heading of 90°.

The speed of sound (in the water), the sampling period, the standard deviations and the number of measurements are chosen to be $c=1500 m/s$, $\Delta t=4s$, $\sigma_D=0.5°$, $\sigma=0.5°$, and $N=225$. The performance obtained with 500 Monte Carlo simulations is summarized in Table 1 and the estimated positions (at initial time), together with the 90% confidence ellipses (at initial and final times), are depicted in Figure 3.

Table 1. Performance of the MLE in the vessel case.

X	Bias	$\hat{\sigma}$	σ_{CRLB}
-2000 (m)	2.3	517.4	532.4
3000 (m)	4.4	778.6	801
5 (m/s)	0.06	1.3	1.33
0 (m/s)	0	0.017	0.016

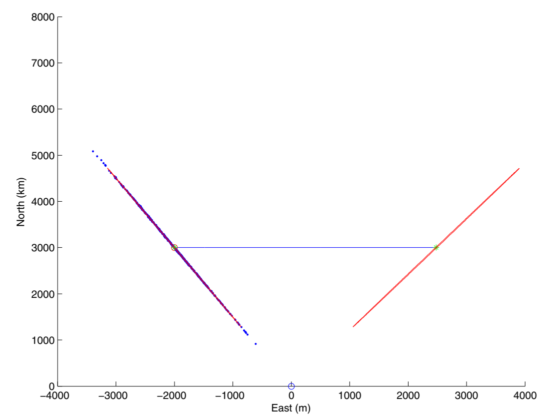

Figure 3. The vessel trajectory, the estimates of the initial position, and the 90% confidence ellipses.

4.3.2 Helicopter

Now, a passive radar and a passive sonar are mounted on the observer. The target's initial position is $P_T(0)=[-300 \quad 2000]^T$ (m). The velocity is equal to $[50 \quad 0]^T$ (m/s).

The number of measurements is $N=10$. The sampling period is chosen to be $\Delta t=1s$. The observer acquires 10 pairs of measurements. The speed of sound (in the air) is $c=330 m/s$. The standard deviations are $\sigma_D=1°$ and $\sigma=1°$. Table 2 presents the performance of the estimator (bias and standard deviation to be compared to those computed with the CRLB). The estimated positions (at initial time) and the 90% confidence ellipses (at initial and final times) are plotted in Figure 4.

Table 2. Performance of the MLE in the helicopter case.

X	Bias	$\hat{\sigma}$	σ_{CRLB}
-300 (m)	0.03	13.92	13.61
2000 (m)	1.4	148.68	155.71
50 (m/s)	0.06	2.8	2.75
0 (m/s)	0.1	27.6	27.5

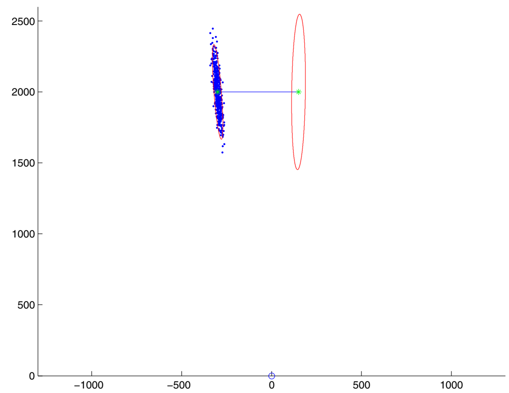

Figure 4(a). The helicopter trajectory, the estimates of the initial position, and the 90% confidence ellipses.

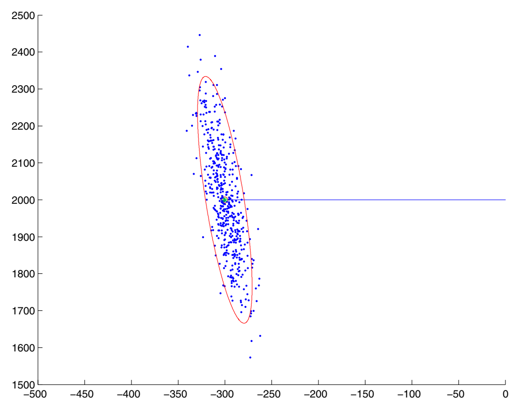

Figure 4(b). Magnification of the confidence ellipse, and the estimates at the initial time.

4.3.3 Airplane

In this last example, the target flies with a speed equal to 150 m/s. The heading is still 90°. Its starts from the point $P_T(0)=[-500 \quad 2000]^T$ (m). Again, the observer collects 10 pairs of measurements only, with the sampling period $\Delta t = 1s$. The standard deviations are $\sigma_D = 1°$ and $\sigma = 1°$, respectively. The performance was evaluated with 500 Monte Carlo runs and is presented in Table 3. The estimated positions (at the initial time) are depicted in Figure 5, together with the 90% confidence ellipses (at initial and final times).

Table 3. Performance of the MLE in the airplane case.

X	Bias	$\hat{\sigma}$	σ_{CRLB}
-500 (m)	0.03	9.25	9.82
2000 (m)	0.35	51.53	51.85
150 (m/s)	0.05	2.48	2.46
0 (m/s)	0.43	8.38	9.17

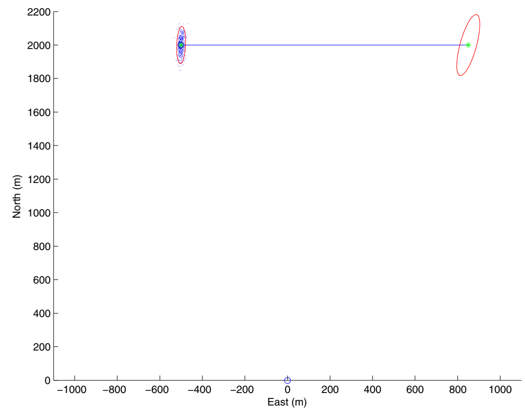

Figure 5(a). The airplane trajectory, the estimates of the initial position, and the 90% confidence ellipses.

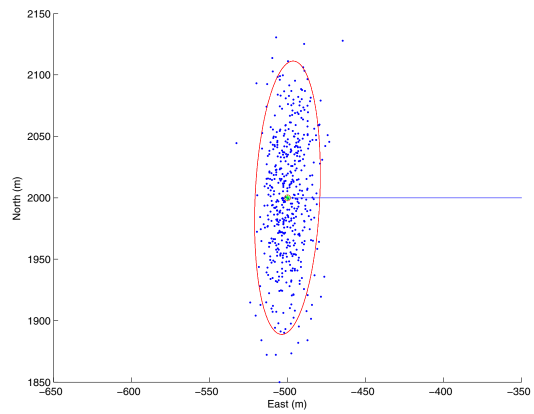

Figure 5(b). Magnification of the confidence ellipse, and the estimates at the initial time.

4.3.4 Comments on these examples

The values presented in Tables 1, 2, and 3 reveal that the MLE is fairly efficient: no significant difference between $\hat{\sigma}$ and σ_{CRLB} is noted. Of course, the performance of this new TMA method is up to a rotation. In any other case, the evaluation of the CRLB can give us prior information about the interest of this technique. At this stage, these three examples are very encouraging to pursue this study.

5 Conclusion

In this paper, we have proposed an original way to passively estimate the trajectory of a source emitting two kinds of waves, each of them propagating with different speeds. In the examples given here, we consider electromagnetic waves and acoustic waves. The asymptotic performance (given by the CRLB) let expect that this method was very interesting. The behavior of the maximum likelihood estimator confirms this, through numerous Monte Carlo simulations. This approach could be extended to the case of acoustic waves only, but propagating in two different media with two different speeds, for example in the air and on the ground for a terrestrial vehicle.

This algorithm is protected under the patent NO 1461970.

Appendix

1. Computation of the time delay

In the following expressions, τ stands for $\tau(t)$:

$$\begin{aligned}
c^2\tau^2 &= R^2(t-\tau) \\
&= \| P_T(0) + (t-\tau)V \|^2 \\
&= \| P_T(0)\|^2 + 2(t-\tau)V^T P_T(0) + (t-\tau)^2 \| V \|^2 \\
&= \| P_T(0)\|^2 + 2(t-\tau)V^T P_T(0) + \left(t^2 - 2t\tau + \tau^2\right)\| V \|^2 \\
&= \| P_T(0)\|^2 + 2tV^T P_T(0) + t^2v^2 - 2\tau V^T P_T(0) - 2t\tau v^2 + \tau^2 v^2 \\
&= \| P_T(0) + tV\|^2 - 2\tau\left(V^T P_T(0) + t\,v^2\right) + \tau^2 v^2
\end{aligned}$$

$$\Leftrightarrow \quad \tau^2\left(c^2 - v^2\right) + 2\tau\left(V^T P_T(0) + t\,v^2\right) - \| P_T(0) + tV\|^2 = 0 .$$

Note that $c^2 - v^2 > 0$ and $\| P_T(0) + tV\|^2 > 0$; the two roots of this equation are of opposite sign. Only the positive root is the value sought:

$$\tau = \frac{-\left(V^T P_T(0) + t\,v^2\right)}{\left(c^2 - v^2\right)} + \frac{\sqrt{\left(V^T P_T(0) + t\,v^2\right)^2 + \left(c^2 - v^2\right)\| P_T(0) + tV\|^2}}{\left(c^2 - v^2\right)},$$

or equivalently

$$\tau(t) = \frac{\sqrt{\left[V^T P_T(t)\right]^2 + \left(c^2 - v^2\right)\| P_T(t)\|^2} - V^T P_T(t)}{\left(c^2 - v^2\right)} .$$

2. Computation of the FIM

Recall that

$$F(X) = \sum_{k=1}^{N} \frac{1}{\sigma^2} \nabla_X \theta(X,t_k) \nabla_X^T \theta(X,t_k) + \sum_{k=1}^{N} \frac{1}{\sigma_D^2} \nabla_X \theta_D(X,t_k) \nabla_X^T \theta_D(X,t_k).$$

We readily obtain

$$\nabla_X \theta(X,t_k) = \frac{1}{R(t_k)}\left[\cos\theta(t_k) \quad -\sin\theta(t_k) \quad t_k\cos\theta(t_k) \quad -t_k\sin\theta(t_k)\right]^T.$$

The difficulty is in computing $\nabla_X \theta_D(X,t_k)$ due to the recursive definition of $\tau(t_k)$.

We denote $\frac{\partial}{\partial e_i}$ the derivative w.r.t. the i[th] component of X, $i=1, 2, 3, 4$:

$$\frac{\partial \theta_D(t_k)}{\partial e_i} = \frac{1}{R_T^2(t_k-\tau(t_k))}\left[y_T(t_k-\tau(t_k))\frac{\partial x_T(t_k-\tau(t_k))}{\partial e_i} - x_T(t_k-\tau(t_k))\frac{\partial y_T(t_k-\tau(t_k))}{\partial e_i}\right]$$

$$= \frac{1}{R_T(t_k-\tau(t_k))}\left[\cos\theta_D(t_k)\frac{\partial x_T(t_k-\tau(t_k))}{\partial e_i} - \sin\theta_D(t_k)\frac{\partial y_T(t_k-\tau(t_k))}{\partial e_i}\right]. \quad \text{(A1)}$$

We now detail each derivative:

$$\frac{\partial x_T(t_k-\tau(t_k))}{\partial x} = 1-\dot{x}\frac{\partial \tau(t_k)}{\partial x} \quad \text{(A2)}$$

$$\frac{\partial y_T(t_k-\tau(t_k))}{\partial x} = -\dot{y}\frac{\partial \tau(t_k)}{\partial x} \quad \text{(A3)}$$

$$\frac{\partial x_T(t_k-\tau(t_k))}{\partial y} = -\dot{x}\frac{\partial \tau(t_k)}{\partial y} \quad \text{(A4)}$$

$$\frac{\partial y_T(t_k-\tau(t_k))}{\partial y} = 1-\dot{y}\frac{\partial \tau(t_k)}{\partial y} \quad \text{(A5)}$$

$$\frac{\partial x_T(t_k-\tau(t_k))}{\partial \dot{x}} = t_k-\tau(t_k)-\dot{x}\frac{\partial \tau(t_k)}{\partial \dot{x}} \quad \text{(A6)}$$

$$\frac{\partial y_T(t_k-\tau(t_k))}{\partial \dot{x}} = -\dot{y}\frac{\partial \tau(t_k)}{\partial \dot{x}} \quad \text{(A7)}$$

$$\frac{\partial x_T(t_k-\tau(t_k))}{\partial \dot{y}} = -\dot{x}\frac{\partial \tau(t_k)}{\partial \dot{y}} \quad \text{(A8)}$$

$$\frac{\partial y_T(t_k-\tau(t_k))}{\partial \dot{y}} = t_k-\tau(t_k)-\dot{y}\frac{\partial \tau(t_k)}{\partial \dot{y}} \quad \text{(A9)}$$

Computation of $\frac{\partial \tau(t_k)}{\partial e_i}$ can be done by using $c^2\,\tau^2(t_k) = x_T^2(t_k-\tau(t_k)) + y_T^2(t_k-\tau(t_k))$. We get

$$c^2\,\tau(t_k)\frac{\partial \tau(t_k)}{\partial e_i} = x_S(t_k-\tau(t_k))\frac{\partial x_T(t_k-\tau(t_k))}{\partial e_i} + y_S(t_k-\tau(t_k))\frac{\partial y_T(t_k-\tau(t_k))}{\partial e_i}.$$

Recalling that $c^2\tau(t_k) = c\,R(t_k-\tau(t_k))$, $x_T(t_k-\tau(t_k)) = R(t_k-\tau(t_k))\sin\theta_D(t_k)$ and $y_T(t_k-\tau(t_k)) = R(t_k-\tau(t_k))\cos\theta_D(t_k)$, we obtain

$$c\frac{\partial \tau(t_k)}{\partial e_i} = \sin\theta_D(t_k)\frac{\partial x_T(t_k-\tau(t_k))}{\partial e_i} + \cos\theta_D(t_k)\frac{\partial y_T(t_k-\tau(t_k))}{\partial e_i}.$$

a) Expression of $\frac{\partial \tau(t_k)}{\partial x}$:

$$c\frac{\partial \tau(t_k)}{\partial x} = \sin\theta_D(t_k)\left[1-\dot{x}\frac{\partial \tau(t_k)}{\partial x}\right] - \cos\theta_D(t_k)\dot{y}\frac{\partial \tau(t_k)}{\partial x}$$

$$\Leftrightarrow \frac{\partial \tau(t_k)}{\partial x}\left(c+\dot{x}\,\sin\theta_D(t_k)+\dot{y}\,\cos\theta_D(t_k)\right) = \sin\theta_D(t_k)$$

$$\Leftrightarrow \frac{\partial \tau(t_k)}{\partial x} = \frac{\sin\theta_D(t_k)}{c+\dot{x}\,\sin\theta_D(t_k)+\dot{y}\,\cos\theta_D(t_k)}. \quad \text{(A10)}$$

b) Expression of $\frac{\partial \tau(t_k)}{\partial y}$:

$$c\frac{\partial \tau(t_k)}{\partial y} = -\sin\theta_D(t_k)\dot{x}\frac{\partial \tau(t_k)}{\partial y} + \cos\theta_D(t_k)\left[1-\dot{y}\frac{\partial \tau(t_k)}{\partial y}\right]$$

$$\Leftrightarrow \frac{\partial \tau(t_k)}{\partial y} = \frac{\cos\theta_D(t_k)}{c+\dot{x}\,\sin\theta_D(t_k)+\dot{y}\,\cos\theta_D(t_k)}. \quad \text{(A11)}$$

c) Expression of $\frac{\partial \tau(t_k)}{\partial \dot{x}}$:

$$c\frac{\partial \tau(t_k)}{\partial \dot{x}} = \sin\theta_D(t_k)\left[t_k-\tau(t_k)-\dot{x}\frac{\partial \tau(t_k)}{\partial \dot{x}}\right] - \cos\theta_D(t_k)\dot{y}\frac{\partial \tau(t_k)}{\partial \dot{x}}$$

$$\Leftrightarrow \frac{\partial \tau(t_k)}{\partial \dot{x}} = \frac{[t_k-\tau(t_k)]\sin\theta_D(t_k)}{c+\dot{x}\,\sin\theta_D(t_k)+\dot{y}\,\cos\theta_D(t_k)} \quad \text{(A12)}$$

d) Expression of $\frac{\partial \tau(t_k)}{\partial \dot{y}}$:

$$c\frac{\partial \tau(t_k)}{\partial \dot{x}} = -\sin\theta_D(t_k)\dot{x}\frac{\partial \tau(t_k)}{\partial \dot{x}} + \cos\theta_D(t_k)\left[t_k-\tau(t_k)-\dot{y}\frac{\partial \tau(t_k)}{\partial \dot{x}}\right]$$

$$\Leftrightarrow \frac{\partial \tau(t_k)}{\partial \dot{y}} = \frac{[t_k-\tau(t_k)]\cos\theta_D(t_k)}{c+\dot{x}\,\sin\theta_D(t_k)+\dot{y}\,\cos\theta_D(t_k)} \quad \text{(A13)}$$

Inserting (A10) - (A13) in (A2) - (A9), we get

$$\frac{\partial x_T(t_k-\tau(t_k))}{\partial x} = 1-\frac{\dot{x}\sin\theta_D(t_k)}{c+\dot{x}\,\sin\theta_D(t_k)+\dot{y}\,\cos\theta_D(t_k)} \quad \text{(A14)}$$

$$\frac{\partial y_T(t_k-\tau(t_k))}{\partial x} = -\frac{\dot{y}\sin\theta_D(t_k)}{c+\dot{x}\,\sin\theta_D(t_k)+\dot{y}\,\cos\theta_D(t_k)} \quad \text{(A15)}$$

$$\frac{\partial x_T(t_k-\tau(t_k))}{\partial y} = -\frac{\dot{x}\cos\theta_D(t_k)}{c+\dot{x}\,\sin\theta_D(t_k)+\dot{y}\,\cos\theta_D(t_k)} \quad \text{(A16)}$$

$$\frac{\partial y_T(t_k-\tau(t_k))}{\partial y} = 1-\frac{\dot{y}\cos\theta_D(t_k)}{c+\dot{x}\,\sin\theta_D(t_k)+\dot{y}\,\cos\theta_D(t_k)} \quad \text{(A17)}$$

$$\frac{\partial x_T(t_k-\tau(t_k))}{\partial \dot{x}}=[t_k-\tau(t_k)]\frac{\partial x_T(t_k-\tau(t_k))}{\partial x} \quad \text{(A18)}$$

$$\frac{\partial y_T(t_k-\tau(t_k))}{\partial \dot{x}}=[t_k-\tau(t_k)]\frac{\partial y_T(t_k-\tau(t_k))}{\partial x} \quad \text{(A19)}$$

$$\frac{\partial x_T(t_k-\tau(t_k))}{\partial \dot{y}}=[t_k-\tau(t_k)]\frac{\partial x_T(t_k-\tau(t_k))}{\partial y} \quad \text{(A20)}$$

$$\frac{\partial y_T(t_k-\tau(t_k))}{\partial \dot{y}}=[t_k-\tau(t_k)]\frac{\partial y_T(t_k-\tau(t_k))}{\partial y} \quad \text{(A21)}$$

Inserting now (A14) – (A21) in (A1), we obtain

$$\frac{\partial \theta_D(t_k)}{\partial x}=\frac{1}{R_T(t_k-\tau(t_k))}\left\{\cos\theta_D(t_k)\right. \\ \left. +\sin\theta_D(t_k)\left[\frac{\dot{y}\sin\theta_D(t_k)-\dot{x}\cos\theta_D(t_k)}{c+\dot{x}\sin\theta_D(t_k)+\dot{y}\cos\theta_D(t_k)}\right]\right\}$$

$$\frac{\partial \theta_D(t_k)}{\partial y}=\frac{1}{R_T(t_k-\tau(t_k))}\left\{-\sin\theta_D(t_k)\right. \\ \left. +\cos\theta_D(t_k)\left[\frac{\dot{y}\sin\theta_D(t_k)-\dot{x}\cos\theta_D(t_k)}{c+\dot{x}\sin\theta_D(t_k)+\dot{y}\cos\theta_D(t_k)}\right]\right\}$$

$$\frac{\partial \theta_D(t_k)}{\partial \dot{x}}=[t_k-\tau(t_k)]\frac{\partial \theta_D(t_k)}{\partial x}$$

$$\frac{\partial \theta_D(t_k)}{\partial \dot{y}}=[t_k-\tau(t_k)]\frac{\partial \theta_D(t_k)}{\partial y}$$

Note that $\lim_{c\to\infty}\frac{\partial \theta_D(t_k)}{\partial e_i}=\frac{\partial \theta(t_k)}{\partial e_i}$.

References

[1] Le Cadre, J.P. and Jauffret, C., "Discrete-Time Observability and Estimability Analysis for Bearings-Only Target Motion Analysis", *IEEE Transactions on Aerospace and Electronic Systems, AES-33*, no. 1, pp. 178-201, January 1997.

[2] Jauffret, C., Pillon, D. and Pignol, A.C., "Leg-by-leg Bearings-Only TMA without Observer Maneuver", *Journal of Advanced Information Fusion,* Vol. 6, no.1, pp. 24-38, June 2011.

[3] Nardone, S.C. and Aidala, V.J., "Observability Criteria for Bearings-Only Target Motion Analysis", *IEEE Transactions on Aerospace and Electronic Systems,* AES-17, 2, pp. 162-166, March 1981.

[4] Jauffret, C., and Pillon, D., "Observability in Passive Target Motion Analysis", *IEEE Transactions on Aerospace and Electronic Systems*, AES-32, 4, pp. 1290-1300, October 1996.

[5] Bar-Shalom, Y., "Update with Out-of-Sequence Measurements in Tracking: Exact Solution", *IEEE Transactions on Aerospace and Electronic Systems*, AES-38, 3, pp. 769-778, July 2002.

[6] Lo, K.W. and Ferguson, B.G., "Broadband Passive Acoustic technique for Target Parameter Estimation", *IEEE Transactions on Aerospace and Electronic Systems*, AES-36, 1, pp. 163-175, January 2000.

[7] Orguner, U. and Gustafsson, F., "Target Tracking Using Delayed Measurements with Implicit Constraints", *Proceeding of 11th International Conference on Information Fusion (FUSION 2008)*, Cologne, Germany, July 2008.

[8] Orguner, U. and Gustafsson, F., "Distributed Target Tracking with Propagation Delayed Measurements", *Proceeding of 12th International Conference on Information Fusion (FUSION 2009)*, Seattle, USA, July 2009.

[9] Orguner, U. and Gustafsson, F., "Target Tracking With Particle Filters Under Signal Propagation Delays", *IEEE Transactions on Signal Processing*, 59, 6, pp. 2485-2495, June 2011.

[10] Bi, S. Z. and Ren. X. Y., "Maneuvering Target Doppler-Bearing Tracking Using Interacting Multiple Models Algorithm", *Progress In Electromagnetics Research*, *PIER* 87, pp. 15-41, 2008.

[11] Guo, Y., Xue. A. and Peng. D, "A Recursive Algorithm for Bearings-Only Tracking with Signal Time Delay", *Signal Processing,* 88, 6, pp. 1539-1552, 2008.

[12] Cheung, B., Davy. S. and Gray. D, "PMHT for Tracking with Timing Uncertainty", *Proceeding of 13th International Conference on Information Fusion (FUSION 2010)*, Edinburg, Scotland, July 2010.