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Tracking Targets With Pairwise-Markov Dynamics

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Abstract* – *Single- and multi-target tracking are both typically based on the* hidden Markov chain *(HMC)* model*. That is, the target process is a Markov chain, observed by an independent observation process. Since HMC independence assumptions are invalid in many practical applications, the* pairwise Markov chain *(PMC) model has been proposed as an approach for weakening them. Petetin and Desbouvries subsequently proposed a PMC generalization of the probability hypothesis density (PHD) filter, but their derivation was somewhat heuristic. The first major purpose of this paper is to construct a solid theoretical foundation for the Petetin-Desbouvries filter—which turns out to be a multitarget HMC model rather than a true multitarget PMC model. The second major purpose is to use this foundation to devise PMC versions of any random finite set (RFS) filter, thus allowing tracking of targets with non-HMC dynamics.

Keywords: Multitarget tracking, PHD filter, pairwise Markov chain, random finite set, point process.

1 Introduction

Single-target and multitarget tracking algorithms are typically based on *hidden Markov chain* (HMC) models. In the single-target case, the evolving target process

$$\mathbf{X}_{1:k} : \mathbf{X}_1, \ldots, \mathbf{X}_k$$

is a Markov chain of random state-vectors that, at times $t_1,\ldots,t_k$ is observed by independent observation processes

$$\mathbf{Y}_{1:k} : \mathbf{Y}_1, \ldots, \mathbf{Y}_k.$$

HMC independence assumptions are often not valid in practice : plant noise can be correlated with measurement noise; or current measurement noise can be correlated with earlier measurement noise (e.g., colored noise).

In the single-target case, Pieczynski's *pairwise Markov chain* (PMC) model [6-9] has been proposed as a means of relaxing HMC assumptions. In a PMC,

$$(\mathbf{X}_{1:k}, \mathbf{Y}_{1:k}) : (\mathbf{X}_1, \mathbf{Y}_1), \ldots, (\mathbf{X}_k, \mathbf{Y}_k)$$

is a Markov chain, even though $\mathbf{X}_{1:k}$ need not be a Markov chain. As a consequence, the following inequalities are possible for a PMC:

$$f(\mathbf{x}_k \mid \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \neq f(\mathbf{x}_k \mid \mathbf{x}_{k-1}), \tag{1}$$

$$f(\mathbf{y}_k \mid \mathbf{x}_k, \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \neq f(\mathbf{y}_k \mid \mathbf{x}_k). \tag{2}$$

Pieczynski and Desbouvries have described concrete, practical Kalman filter-based applications and implementations of PMCs to single-target tracking in [8].

A detailed tutorial introduction to PMCs is beyond the scope of this paper. Readers with questions should consult [8] and the *IEEE Transactions* papers [6,9].

Petetin and Desbouvries proposed a PMC generalization of the probability hypothesis density (PHD) filter in their 2013 *IEEE Transactions* paper [5]. They described concrete practical applications and implementations based on [8]; and demonstrated that their filter has better tracking performance than the classical PHD filter under non-HMC conditions.

However, their derivation was somewhat heuristic in that it lacked a general theoretical foundation. This paper therefore has two major purposes. The first is to provide such a foundation. As will be seen, this turns out to be a multitarget HMC (M-HMC) model rather than a true multitarget PMC (M-PMC) model. That is, an M-HMC model is used to address targets that have, individually, PMC dynamics. The crucial insight consists of Eqs. (37,38), which show that any PMC can be reformulated as an HMC with a Dirac-delta likelihood function.

The second major purpose is to use this M-HMC foundation to allow the construction of PMC versions of any random finite set (RFS) filter, thus making it generally possible to track targets with PMC dynamics.

However, this denouement leaves a major question left unaddressed: What of multitarget systems that are governed by true M-PMC dynamics? An investigation into this matter was initiated in the recent conference paper [2]. It was shown there that the PMC approach can be directly generalized to multitarget detection and tracking (see Section 3)—but that the resulting multitarget filters appear to be computationally intractable. Thus [2] also reported an exploratory attempt to devise, using finite-set statistics [1,3,4] techniques, a cardinalized PHD (CPHD) filter predicated on true M-PMC dynamics.

This paper is organized as follows. Single-target PMCs are reviewed in Section 2 and their generalization to multitarget PMCs in Section 3. The Petetin-Desbouvries PMC-PHD filter is briefly reviewed in Section 4. The theory for this and other PMC-RFS filters is described in Section 5. As an example, the formulas for a PMC-CPHD filter are given in Section 6. Mathematical derivations have been relegated to Section 7. Conclusions can be found in Section 8.

Since Petetin and Desbouvries initiated the study of M-PMC systems, their PMC notation will be employed throughout. This includes their practice of distinguishing between two kinds of measurements: unknown measurements $\mathbf{y}_k$ that are part of a PMC state $(\mathbf{x}_k,\mathbf{y}_k)$ and known measurements $\mathbf{z}_k$ collected from that state.

2 Single-Target PMCs

HMCs are the theoretical basis for conventional single-sensor, single-target tracking. Let $\mathbf{x}_k$ be the state of a target at time t_k. Assume that: (1) the time-evolution of $\mathbf{x}_k$ is described by a Markov transition density $f(\mathbf{x}_k|\mathbf{x}_{k-1})$; (2) the target is observed by a single sensor with unity probability of detection and no false alarms; and (3) $f(\mathbf{z}_k|\mathbf{x}_k)$ is the probability (density) that measurement $\mathbf{z}_k$ will be collected if the target has state $\mathbf{x}_k$. Then the stochastic dynamical system $(\mathbf{X}_k, \mathbf{Y}_k)$ for $k \geq 1$ is an HMC if its total bivariate distribution $f(\mathbf{x}_{1:k}|\mathbf{y}_{1:k})$ factors as (see Eq. (1) of [5]):

$$f(\mathbf{x}_{1:k},\mathbf{y}_{1:k}) = f(\mathbf{x}_1)\left(\prod_{i=2}^{k} f(\mathbf{x}_i|\mathbf{x}_{i-1})\right)\left(\prod_{i=1}^{k} f(\mathbf{y}_i|\mathbf{x}_i)\right) \tag{3}$$

where $f(\mathbf{x}_1)$ is the distribution of the target state at time t_1. Equivalently, the HMC is defined by the recursion

$$f(\mathbf{x}_{1:k},\mathbf{y}_{1:k}) = f(\mathbf{x}_{1:k-1},\mathbf{y}_{1:k-1}) \cdot f(\mathbf{x}_k|\mathbf{x}_{k-1})\cdot f(\mathbf{y}_k|\mathbf{x}_k) \tag{4}$$

for $k > 1$. Integrating both sides of this with respect to $\mathbf{x}_{1:k-1}$ results in

$$f(\mathbf{x}_k,\mathbf{y}_{1:k}) = f(\mathbf{y}_k|\mathbf{x}_k)\int f(\mathbf{x}_k|\mathbf{x}_{k-1}) \cdot f(\mathbf{x}_{k-1},\mathbf{y}_{1:k-1})d\mathbf{x}_{k-1} \tag{5}$$

from which follows the single-equation recursion for the single-target Bayes filter:

$$f(\mathbf{x}_k|\mathbf{y}_{1:k}) = f(\mathbf{y}_k|\mathbf{y}_{1:k-1})^{-1}\cdot f(\mathbf{y}_k|\mathbf{x}_k) \cdot\int f(\mathbf{x}_k|\mathbf{x}_{k-1})\cdot f(\mathbf{x}_{k-1}|\mathbf{y}_{1:k-1})d\mathbf{x}_{k-1}. \tag{6}$$

PMCs were introduced in 2000 by Pieczynski [6,9] and have begun to generate a substantial sub-literature. A PMC is a joint dynamical system $(\mathbf{X}_{1:k}, \mathbf{Y}_{1:k})$ whose evolution is governed by a bivariate Markov transition density

$$f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1}) = f(\mathbf{y}_k|\mathbf{x}_k,\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \cdot f(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \tag{7}$$

(where the right side follows from Bayes' rule) and whose total bivariate distribution factors as

$$f(\mathbf{x}_{1:k},\mathbf{y}_{1:k}) = f(\mathbf{x}_1,\mathbf{y}_1)\prod_{i=2}^{k} f(\mathbf{x}_i,\mathbf{y}_i|\mathbf{x}_{i-1},\mathbf{y}_{i-1}), \tag{8}$$

where $f(\mathbf{x}_1,\mathbf{y}_1)$ is the distribution at time t_1.

PMCs significantly weaken HMC independence assumptions—see, for example, [5], p. 4487. Measurement and plant noises need not be uncorrelated, for example; and measurement noise need not be uncorrelated frame-to-frame.

A PMC reduces to an HMC if

$$f(\mathbf{x}_1,\mathbf{y}_1) = f(\mathbf{x}_1)\cdot f(\mathbf{y}_1|\mathbf{x}_1) \tag{9}$$

and if, for $k > 1$,

$$f(\mathbf{y}_k|\mathbf{x}_k,\mathbf{x}_{k-1},\mathbf{y}_{k-1}) = f(\mathbf{y}_k|\mathbf{x}_k), \tag{10}$$

$$f(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1}) = f(\mathbf{x}_k|\mathbf{x}_{k-1}). \tag{11}$$

A target can be tracked even if its state is part of a PMC but is not itself an HMC. The recursion formula for the measurement-updated target state is (see Eq. (12) of Petetin and Desbouvries' *IEEE Transactions* paper [5]):

$$f(\mathbf{x}_k|\mathbf{y}_{1:k}) = \frac{\int f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})\cdot f(\mathbf{x}_{k-1}|\mathbf{y}_{1:k-1})d\mathbf{x}_{k-1}}{\int f(\mathbf{x}'_k,\mathbf{y}_k|\mathbf{x}'_{k-1},\mathbf{y}_{k-1})\cdot f(\mathbf{x}'_{k-1}|\mathbf{y}_{1:k-1})d\mathbf{x}'_{k-1}d\mathbf{x}'_k}. \tag{12}$$

From this, the recursion formula for the measurement-updated PMC state $(\mathbf{x}_k,\mathbf{y}_k)$ can be shown to be:

$$f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{y}_{1:k-1}) = \frac{\int f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})\cdot f(\mathbf{x}_{k-1},\mathbf{y}_{k-1}|\mathbf{y}_{1:k-2})d\mathbf{x}_{k-1}}{\int f(\mathbf{x}'_{k-1},\mathbf{y}_{k-1}|\mathbf{y}_{1:k-2})d\mathbf{x}'_{k-1}}. \tag{13}$$

For completeness, the derivations of Eqs. (12,13) are given in Section 7. The two distributions are related by

$$f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{y}_{1:k-1}) = f(\mathbf{y}_k|\mathbf{y}_{1:k-1})\cdot f(\mathbf{x}_k|\mathbf{y}_{1:k}) \tag{14}$$

where

$$f(\mathbf{y}_k|\mathbf{y}_{1:k-1}) = \int f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}|\mathbf{y}_{1:k-1})d\mathbf{x}_{k-1}d\mathbf{x}_k. \tag{15}$$

3 Multitarget PMCs (M-PMCs)

An M-PMC generalizes a PMC in the obvious manner. Let $X = \{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ be a multitarget state-set with $|X| = n \geq 0$ and let $Y = \{\mathbf{y}_1,\ldots,\mathbf{y}_m\}$ be a multitarget measurement-set with $|Y| = m \geq 0$. An M-HMC is defined by the factorization

$$f(X_{1:k},Y_{1:k}) = f(X_1)\left(\prod_{i=2}^{k} f(X_i|X_{i-1})\right)\left(\prod_{i=1}^{k} f(Y_i|X_i)\right) \tag{16}$$

where $f(X_i|X_{i-1})$ is the multitarget Markov transition density and $f(Z_i|X_i)$ is the single-sensor, multitarget likelihood function. The multitarget analog of Eq. (6) is the multitarget Bayes filter [1,3,4]:

$$f(X_k \mid Y_{1:k}) = f(Y_k \mid Y_{1:k-1})^{-1} \cdot f(Y_k \mid X_k) \cdot \int f(X_k \mid X_{k-1}) \cdot f(X_{k-1} \mid Y_{1:k-1}) \delta X_{k-1} \quad (17)$$

where the indicated integral is a set integral.

A multitarget PMC is defined by the factorization

$$f(X_{1:k}, Y_{1:k}) = f(X_1, Y_1) \prod_{i=2}^{k} f(X_i, Y_i | X_{i-1}, Y_{i-1}), \quad (18)$$

where

$$f(X_k, Y_k | X_{k-1}, Y_{k-1}) = f(Y_k | X_k, X_{k-1}, Y_{k-1}) \cdot f(X_k | X_{k-1}, Y_{k-1}) \quad (19)$$

is the transition density that specifies M-PMC evolution.

The obvious multitarget analog of the multitarget recursion of Eq. (12) is

$$f(X_k \mid Y_{1:k}) = \frac{\int f(X_k, Y_k \mid X_{k-1}, Y_{k-1}) \cdot f(X_{k-1} \mid Y_{1:k-1}) \delta X_{k-1}}{\int f(X_k', Y_k \mid X_{k-1}', Y_{k-1}) \cdot f(X_{k-1}' \mid Y_{1:k-1}) \delta X_{k-1}' \delta X_k'} \quad (20)$$

The obvious multitarget analog of the PMC recursion, Eq. (13), is

$$f(X_k, Y_k \mid Y_{1:k-1}) = \frac{\int f(X_k, Y_k \mid X_{k-1}, Y_{k-1}) \cdot f(X_{k-1}, Y_{k-1} \mid Y_{1:k-2}) \delta X_{k-1}}{\int f(X_{k-1}', Y_{k-1} \mid Y_{1:k-2}) \delta X_{k-1}'} \quad (21)$$

where the integrals are set integrals. The two distributions are related by

$$f(X_k, Y_k | Y_{1:k-1}) = f(Y_k | Y_{1:k-1}) \cdot f(X_k | Y_{1:k}) \quad (22)$$

where

$$f(Y_k \mid Y_{1:k-1}) = \int f(X_k, Y_k \mid X_{k-1}, Y_{k-1}) \cdot f(X_{k-1} \mid Y_{1:k-1}) \delta X_{k-1} \delta X_k. \quad (23)$$

4 The Petetin-Desbouvries Filter

Because the multitarget Bayes filter of Eq. (17) is computationally intensive in general, the techniques of finite-set statistics have been used to derive various approximate filters—see [1,3,4]. The simplest of these is the PHD filter, in which the predicted multitarget distributions $f(X_k|Z_{1:k-1})$ are assumed to be approximately Poisson:

$$f(X_k|Z_{1:k-1}) = e^{-\int D(\mathbf{x}_k|Z_{1:k-1}) d\mathbf{x}_k} \prod_{\mathbf{x} \in X_k} D(\mathbf{x}_k|Z_{1:k-1}). \quad (24)$$

The PHD filter recursion (without spawning) is given by

$$D(\mathbf{x}_k \mid Z_{1:k-1}) = b(\mathbf{x}_k) + \int p_S(\mathbf{x}_{k-1}) \cdot f(\mathbf{x}_k \mid \mathbf{x}_{k-1}) \cdot D(\mathbf{x}_{k-1} \mid Z_{1:k-1}) d\mathbf{x}_{k-1} \quad (25)$$

and

$$\frac{D(\mathbf{x}_k \mid Z_{1:k})}{D(\mathbf{x}_k \mid Z_{1:k-1})} = 1 - p_D(\mathbf{x}_k) + \sum_{\mathbf{z}_k \in Z_k} \frac{p_D(\mathbf{x}_k) \cdot f(\mathbf{y}_k \mid \mathbf{x}_k)}{\kappa(\mathbf{z}_k) + \tau(\mathbf{z}_k \mid Z_{1:k-1})} \quad (26)$$

where

$$\tau(\mathbf{z}_k \mid Z_{1:k-1}) = \int p_D(\mathbf{x}_k) \cdot f(\mathbf{y}_k \mid \mathbf{x}_k) \cdot D(\mathbf{x}_k \mid Z_{1:k-1}) d\mathbf{x}_k. \quad (27)$$

Here, $b(\mathbf{x}_k)$ is the PHD for target appearances; $p_S(\mathbf{x}_{k-1})$ is the survival probability for a target with state $\mathbf{x}_{k-1}$; $p_D(\mathbf{x}_k)$ is the probability of detection for a target with state $\mathbf{x}_k$; $\kappa(\mathbf{z}_k)$ is the PHD (intensity function) of the clutter process; and Z_k is the measurement-set at time t_k.

Since the PMC recursion equations Eqs. (20,21) are computationally demanding, a PHD filter-like approximation of one or both would be even more desirable. Petetin and Desbouvries have proposed such a filter in their *IEEE Transactions* paper with time-update equation ([5], Eq. (20))

$$D(\mathbf{x}_k, \mathbf{y}_k \mid Z_{1:k-1}) = \tilde{b}_k(\mathbf{x}_k, \mathbf{y}_k) + \int p_S(\mathbf{x}_{k-1}) \cdot f(\mathbf{x}_k, \mathbf{y}_k \mid \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot D(\mathbf{x}_{k-1}, \mathbf{y}_{k-1} \mid Z_{1:k-1}) d\mathbf{x}_{k-1} d\mathbf{y}_{k-1} \quad (28)$$

and measurement-update equation ([5], Eq. (21))

$$\frac{D(\mathbf{x}_k, \mathbf{y}_k \mid Z_{1:k})}{D(\mathbf{x}_k, \mathbf{y}_k \mid Z_{1:k-1})} = 1 - p_D(\mathbf{x}_k) + \sum_{\mathbf{z}_k \in Z_k} \frac{p_D(\mathbf{x}_k) \cdot \delta_{\mathbf{z}_k}(\mathbf{y}_k)}{\kappa(\mathbf{z}_k) + \tau(\mathbf{z}_k \mid Z_{1:k-1})} \quad (29)$$

where

$$\tau(\mathbf{z}_k \mid Z_{1:k-1}) = \int p_D(\mathbf{x}_k) \cdot D(\mathbf{x}_k, \mathbf{z}_k \mid Z_{1:k-1}) d\mathbf{x}_k. \quad (30)$$

Furthermore, suppose that single-target PMC dynamics are actually HMC. That is, assume Eqs. (10,11) and

$$\tilde{b}(\mathbf{x}_k, \mathbf{y}_k) = b(\mathbf{x}_k) \cdot f(\mathbf{y}_k \mid \mathbf{x}_k). \quad (31)$$

Then Petetin and Desbouvries note that Eqs. (28,29) reduce to Eqs. (25,26) if they are marginalized by integrating with respect to $\mathbf{y}_k$. For, after marginalization Eq. (28) becomes

$$D(\mathbf{x}_k \mid Z_{1:k-1}) = b_k(\mathbf{x}_k) + \int p_S(\mathbf{x}_{k-1}) \cdot f(\mathbf{x}_k \mid \mathbf{x}_{k-1}) \cdot D(\mathbf{x}_{k-1} \mid Z_{1:k-1}) d\mathbf{x}_{k-1} \quad (32)$$

and, likewise, Eq. (29) becomes

$$D(\mathbf{x}_k, \mid Z_{1:k}) = (1 - p_D(\mathbf{x}_k)) \cdot D(\mathbf{x}_k \mid Z_{1:k-1}) + \sum_{\mathbf{z}_k \in Z_k} \frac{p_D(\mathbf{x}_k) \cdot D(\mathbf{x}_k, \mathbf{z}_k \mid Z_{1:k-1})}{\kappa(\mathbf{z}_k) + \tau(\mathbf{z}_k \mid Z_{1:k-1})} \quad (33)$$

where, after some algebra,

$$D(\mathbf{x}_k, \mathbf{z}_k \mid Z_{1:k-1}) = f(\mathbf{z}_k \mid \mathbf{x}_k)$$

$$\cdot\begin{pmatrix} b(\mathbf{x}_k)+\int p_S(\mathbf{x}_{k-1})\cdot f(\mathbf{x}_k \mid \mathbf{x}_{k-1}) \\ \cdot D(\mathbf{x}_{k-1} \mid Z_{1:k-1})d\mathbf{x}_{k-1} \end{pmatrix} \quad (34)$$

$$= f(\mathbf{z}_k \mid \mathbf{x}_k)\cdot D(\mathbf{x}_k \mid Z_{1:k-1}) \quad (35)$$

in which case Eq. (33) becomes equal to Eq. (26) since Eq. (30) similarly becomes equal to Eq. (27).

5 M-HMC Theory for PMC Targets

The section is organized as follows: An HMC formulation of PMCs (Section 5.1); an M-HMC formulation of M-PMCs (Section 5.2); and a theoretical M-HMC foundation for the RFS filtering of PMC targets (Section 5.3).

5.1 An HMC Formulation of PMCs

A PMC can be reformulated as a particular kind of HMC. Regard the Markov chain $(\mathbf{X}_{1:k}, \mathbf{Y}_{1:k})$ as an HMC, observed with a sequence $\mathbf{z}_{1:k}$ of collected measurements. That is, the state of the PMC system at time t_k is $(\mathbf{x}_k, \mathbf{y}_k)$ and a measurement $\mathbf{z}_k$ is collected from it. Define the likelihood function

$$L_{\mathbf{z}_k}(\mathbf{x}_k,\mathbf{y}_k) = f(\mathbf{z}_k|\mathbf{x}_k,\mathbf{y}_k) = \delta_{\mathbf{z}_k}(\mathbf{y}_k) \quad (36)$$

where $\delta_{\mathbf{z}}(\mathbf{y})$ is the Dirac delta function concentrated at $\mathbf{z}$. Then Eq. (13) is equivalent to a recursive Bayes filter with time-update equation

$$\begin{aligned} &f(\mathbf{x}_k,\mathbf{y}_k \mid \mathbf{z}_{1:k-1}) \\ &= \int f(\mathbf{x}_k,\mathbf{y}_k \mid \mathbf{x}_{k-1},\mathbf{y}_{k-1}) \\ &\quad \cdot f(\mathbf{x}_{k-1},\mathbf{y}_{k-1} \mid \mathbf{z}_{1:k-1})d\mathbf{x}_{k-1}d\mathbf{y}_{k-1} \end{aligned} \quad (37)$$

and measurement-update equation

$$\begin{aligned} &f(\mathbf{x}_k,\mathbf{y}_k \mid \mathbf{z}_{1:k}) \\ &= \frac{L_{\mathbf{z}_k}(\mathbf{x}_k,\mathbf{y}_k)\cdot f(\mathbf{x}_k,\mathbf{y}_k \mid \mathbf{z}_{1:k-1})}{\int L_{\mathbf{z}_k}(\mathbf{x}'_k,\mathbf{y}'_k)\cdot f(\mathbf{x}'_k,\mathbf{y}'_k \mid \mathbf{z}_{1:k-1})d\mathbf{x}'_k d\mathbf{y}'_k}. \end{aligned} \quad (38)$$

For, from Eq. (38) we get

$$\begin{aligned} &f(\mathbf{x}_{k-1},\mathbf{y}_{k-1} \mid \mathbf{z}_{1:k-1}) \\ &= \frac{\delta_{\mathbf{z}_{k-1}}(\mathbf{y}_{k-1})\cdot f(\mathbf{x}_{k-1},\mathbf{y}_{k-1} \mid \mathbf{z}_{1:k-2})}{\int \delta_{\mathbf{z}_{k-1}}(\mathbf{y}'_{k-1})\cdot f(\mathbf{x}'_{k-1},\mathbf{y}'_{k-1} \mid \mathbf{z}_{1:k-2})d\mathbf{x}'_{k-1}d\mathbf{y}'_{k-1}} \end{aligned} \quad (39)$$

Substitute this into Eq. (37) to get Eq. (13):

$$\begin{aligned} &f(\mathbf{x}_k,\mathbf{y}_k \mid \mathbf{z}_{1:k-1}) \\ &= \frac{\int f(\mathbf{x}_k,\mathbf{y}_k \mid \mathbf{x}_{k-1},\mathbf{z}_{k-1})\cdot f(\mathbf{x}_{k-1},\mathbf{z}_{k-1} \mid \mathbf{z}_{1:k-2})d\mathbf{x}_{k-1}}{\int f(\mathbf{x}'_{k-1},\mathbf{z}'_{k-1} \mid \mathbf{z}_{1:k-2})d\mathbf{x}'_{k-1}}. \end{aligned} \quad (40)$$

5.2 An M-HMC Formulation of M-PMCs

An M-PMC can be similarly recast as an M-HMC, via the following analogs of Eqs. (37,38):

$$\begin{aligned} f(X_k,Y_k \mid Z_{1:k-1}) = &\int f(X_k,Y_k \mid X_{k-1},Y_{k-1}) \\ &\cdot f(X_{k-1},Y_{k-1} \mid Z_{1:k-1})\delta X_{k-1}\delta Y_{k-1} \end{aligned} \quad (41)$$

and

$$\begin{aligned} &f(X_k,Y_k \mid Z_{1:k}) \\ &= \frac{L_{Z_k}(X_k,Y_k)\cdot f(X_k,Y_k \mid Z_{1:k-1})}{\int L_{\mathbf{z}_k}(X'_k,Y'_k)\cdot f(X'_k,Y'_k \mid Z_{1:k-1})\delta X'_k\delta Y'_k}, \end{aligned} \quad (42)$$

and where

$$L_Z(X,Y) = \delta_Z(Y) \quad (43)$$

and where $\delta_Z(Y)$ is the multi-object Dirac delta function concentrated at Z, as defined in [1], Eq. (4.15).

5.3 A Theory of Multiple PMC Targets

Eqs. (37,38) provide the crucial insight necessary for a systematic theoretical foundation for the Petetin-Desbouvries filter. A single-sensor, single-target HMC is a Markov chain $\mathbf{X}_{1:k}$, equipped with a transition density $f(\mathbf{x}_k|\mathbf{x}_{k-1})$ and a likelihood function $f(\mathbf{z}_k|\mathbf{x}_k)$. A single-sensor, single-target PMC is a Markov chain $(\mathbf{X}_{1:k}, \mathbf{Y}_{1:k})$, equipped with a transition density $f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})$ and a special Dirac-delta likelihood function $f(\mathbf{z}_k|\mathbf{x}_k,\mathbf{y}_k)$. PMC theory is extended to the multi-object case using an analogy with conventional RFS multitarget tracking. Make the following assumptions :

1) The evolution of a system of PMC pairs is described by a Markov chain $\dot{\Xi}_{1:k}$ of RFSs $\dot{\Xi}_i$, where any draw $\dot{\Xi}_i = \dot{X}_i$ of $\dot{\Xi}_i$ is a finite set of PMC pairs.
2) The state-transition of an individual PMC pair $(\mathbf{x}_{k-1},\mathbf{y}_{k-1})$ is a Bernoulli process. That is, either $(\mathbf{x}_{k-1},\mathbf{y}_{k-1})$ disappears with probability $1-p_S(\mathbf{x}_{k-1},\mathbf{y}_{k-1})$ or else it survives with probability $p_S(\mathbf{x}_{k-1},\mathbf{y}_{k-1})$, in which case its dynamics are described by the PMC transition function $f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})$.
3) The probability of survival of a PMC pair is that of the corresponding target: $p_S(\mathbf{x}_{k-1},\mathbf{y}_{k-1}) = p_S(\mathbf{x}_{k-1})$.
4) The state-transition processes of multiple PMC pairs are statistically independent, in which case the multi-object transition process is multi-Bernoulli.
5) In any such state transition, new PMC pairs may spontaneously and independently appear.
6) The measurements collected from a system of PMC pairs at time t_k is described by an RFS Σ_k, where any instantiation $\Sigma_k = Z_k$ of Σ_k is a finite set of measurements .
7) The measurement-generation process for an individual pair $(\mathbf{x}_k,\mathbf{y}_k)$ is Bernoulli. That is, either $(\mathbf{x}_k,\mathbf{y}_k)$ generates no measurement with probability $1-p_D(\mathbf{x}_k,\mathbf{y}_k)$, or else it generates a measurement with probability $p_D(\mathbf{x}_k,\mathbf{y}_k)$, in which case the generated measurement is $\mathbf{y}_k$.
8) The probability of detection for a PMC pair is that of the corresponding target: $p_D(\mathbf{x}_k,\mathbf{y}_k), = p_D(\mathbf{x}_k)$.
9) The measurement processes of multiple PMC pairs are statistically independent, in which case the multi-object measurement process is multi-Bernoulli.
10) During any measurement collection, spurious clutter measurements may independently occur.

The consequence of these assumptions is that a system of multiple PMC pairs is a multitarget HMC, not a true multitarget PMC. Because of Assumptions 1-5, the multi-object Markov state-transition density for this system is given by Eq. (7.6) of [1]:

$$\dot{f}(\dot{X}_k \mid \dot{X}_{k-1}) = \dot{b}(\dot{X}_k)\cdot(1-\dot{p}_S)^{\dot{X}_{k-1}} \cdot\sum_{\theta}\prod_{i\,:\,\theta(i)>0}\frac{\left(\begin{array}{c}\dot{p}_S(\mathbf{x}_{k-1,i},\mathbf{y}_{k-1,i})\\ \cdot f(\mathbf{x}_{k,\theta(i)},\mathbf{y}_{k,\theta(i)}\mid \mathbf{x}_{k-1i},\mathbf{y}_{k-1i})\end{array}\right)}{\left(\begin{array}{c}(1-\dot{p}_S(\mathbf{x}_{k-1,i},\mathbf{y}_{k-1,i}))\\ \cdot\dot{b}(\mathbf{x}_{k,\theta(i)},\mathbf{y}_{k,\theta(i)})\end{array}\right)} \tag{44}$$

with corresponding probability generating functional (p.g.fl.) given by [1], Eq. (7.5)

$$\dot{G}[\dot{h}_k|\dot{X}_{k-1}] = \dot{G}^B[\dot{h}_k]\cdot(1-\dot{p}_S+\dot{p}_S M_{\dot{h}_k})^{\dot{X}_{k-1}}. \tag{45}$$

The consequence of Assumptions 6-10 is that the multi-object likelihood function is given by Eq. (7.2) of [4]:

$$f(Z_k \mid \dot{X}_k) = \kappa(Z_k)\cdot(1-\dot{p}_D)^{\dot{X}_k} \cdot\sum_{\theta}\prod_{i\,:\,\theta(i)>0}\frac{\dot{p}_D(\mathbf{x}_{k,i},\mathbf{y}_{k,i})\cdot\delta_{\mathbf{z}_{k,\theta(i)}}(\mathbf{y}_{k,i})}{(1-\dot{p}_D(\mathbf{x}_{k,i},\mathbf{y}_{k,i}))\cdot\kappa(\mathbf{z}_{k,\theta(i)})} \tag{46}$$

with corresponding p.g.fl. given by [1], Eq. (7.1):

$$G[g_k \mid \dot{X}_k] = G^{\kappa}[g_k]\cdot(1-\dot{p}_D+\dot{p}_D\dot{L}_{g_k})^{\dot{X}_k}, \tag{47}$$

$$\dot{L}_{g_k}(\mathbf{x}_{i,i},\mathbf{y}_{k,i}) = g_k(\mathbf{y}_{k,i}). \tag{48}$$

From Eq. (10,11) it follows that the corresponding M-PMC transition function is $f(Y_k|X_k)\cdot f(X_k|X_{k-1})$.

It also follows from this discussion that any RFS filter can, if it is based on standard multitarget tracking assumptions, be transformed into an M-PMC filter by simply substituting $f(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})$ for $f(\mathbf{x}_k|\mathbf{x}_{k-1})$, $f(\mathbf{z}_k|\mathbf{x}_k,\mathbf{y}_k)$ for $f(\mathbf{z}_k|\mathbf{x}_k)$, $p_S(\mathbf{x}_{k-1},\mathbf{y}_{k-1})$ for $p_S(\mathbf{x}_{k-1})$, and $p_D(\mathbf{x}_k,\mathbf{y}_k)$ for $p_D(\mathbf{x}_k)$. The clutter process remains unchanged, whereas the birth process must be defined for PMC pairs $(\mathbf{x}_k,\mathbf{y}_k)$ rather than for targets $\mathbf{x}_k$.

Furthermore, it follows that—from a strictly rigorous point of view---the Petetin-Desbouvries approach is implicitly an H-PMC constructed from single-target PMCs. A true M-PMC would require the construction of an M-PMC transition function $f(X_k,Y_k|X_{k-1},Y_{k-1})$. Preferably, in the M-HMC special case this transition function should reduce to $f(X_k|X_{k-1})\cdot f(Y_k|X_k)$, where $f(Y_k|X_k)$ and $f(X_k|X_{k-1})$ are, respectively, the standard multitarget likelihood function and standard multitarget transition density. This was the subject of the recent conference paper [2].

6 Example : A PMC-CPHD Filter

The section is organized as follows: The PMC-CPHD filter time-update (Section 6.1), the PMC-CPHD filter measurement-update (Section 6.2), and multitarget state estimation (Section 6.3).

6.1 Time-Update

Suppose that we are given: (1) the spatial distribution

$$\dot{s}(\mathbf{x}_{k-1}|Z_{1:k-1});$$

the expected number of PMC pairs

$$\dot{N}_{k-1};$$

and the distribution

$$\dot{p}(\dot{n}_{k-1}|Z_{1:k-1})$$

on the number

$$\dot{n}_{k-1}$$

of PMC pairs (i.e., cardinality distribution) or, equivalently, its probability generating function (p.g.f.)

$$\dot{G}(\dot{x}_{k-1}|Z_{1:k-1}).$$

Given this, the time-updated PHD is given by

$$\dot{D}(\mathbf{x}_k,\mathbf{y}_k \mid Z_{1:k-1}) = \dot{b}(\mathbf{x}_k,\mathbf{y}_k) + \int \dot{p}_S(\mathbf{x}_{k-1},\mathbf{y}_{k-1})\cdot f(\mathbf{x}_k,\mathbf{y}_k\mid\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \cdot\dot{D}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}\mid Z_{1:k-1})d\mathbf{x}_{k-1}d\mathbf{y}_{k-1}. \tag{49}$$

The corresponding spatial distribution is

$$\dot{s}(\mathbf{x}_k,\mathbf{y}_k|Z_{1:k-1}) = \frac{\dot{D}(\mathbf{x}_k,\mathbf{y}_k|Z_{1:k-1})}{\int\dot{D}(\mathbf{x}'_k,\mathbf{y}'_k|Z_{1:k-1})\mathbf{x}'_k d\mathbf{y}'_k}. \tag{50}$$

The predicted cardinality distribution and its corresponding p.g.f. are given by

$$\dot{G}(\dot{x}_k|Z_{1:k-1}) = \dot{G}^B(\dot{x}_k)\cdot\dot{G}(1-\dot{\psi}_k+\dot{\psi}_k\cdot\dot{x}_k|Z_{1:k-1})$$

$$\dot{p}(\dot{n}_k|Z_{1:k-1}) = \sum_{\dot{n}_{k-1}\geq 0}\dot{p}(\dot{n}_k|\dot{n}_{k-1})\cdot\dot{p}(\dot{n}_{k-1}|Z_{1:k-1}) \tag{51}$$

where

$$\dot{\psi}_k = \int\dot{p}_S(\mathbf{x}_{k-1},\mathbf{y}_{k-1})\cdot\dot{s}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}|Z_{1:k-1})\mathbf{x}_{k-1}d\mathbf{y}_{k-1} \tag{52}$$

and where

$$\dot{p}(\dot{n}_k|\dot{n}_{k-1}) = \sum_{i=0}^{\dot{n}_k}p^B(\dot{n}_k-i)\cdot C_{\dot{n}_{k-1},i}\cdot\dot{\psi}_k^i(1-\dot{\psi}_k)^{\dot{n}_{k-1}-i}. \tag{53}$$

6.2 Measurement- Update

Suppose that we have: (1) a new measurement-set

$$Z_k = \{\mathbf{z}_1,\ldots,\mathbf{z}_{m_k}\}$$

with $|Z_k| = m_k$; the predicted spatial distribution

$$\dot{s}(\mathbf{x}_k,\mathbf{y}_k|Z_{1:k-1});$$

and the predicted cardinality distribution

$$\dot{p}(\dot{n}_k|Z_{1:k-1})$$

or, equivalently, its p.g.f.

$$\dot{G}(\dot{x}_k|Z_{1:k-1})$$

(which will be abbreviated as

$$\dot{G}(\dot{x}_k)$$).

Let

$$\dot{N}_k = \dot{G}^{(1)}(1|Z_{1:k-1}). \tag{55}$$

Given this, the predicted PHD is given by the PHD is given by

$$\dot{D}(\mathbf{x}_k, \mathbf{y}_k \mid Z_{1:k}) = \dot{L}_{Z_k}(\mathbf{x}_k, \mathbf{y}_k) \cdot \dot{D}(\mathbf{x}_k, \mathbf{y}_k \mid Z_{1:k-1}) \tag{56}$$

where

$$\dot{L}_{Z_k}(\mathbf{x}_k, \mathbf{y}_k) = \frac{1}{N_k} \cdot \left(\begin{array}{c} (1 - \dot{p}_D(\mathbf{x}_k, \mathbf{y}_k)) \cdot \overset{\text{ND}}{L}_{Z_k} \\ + \sum_{j=1}^{m_k} \frac{\dot{p}_D(\mathbf{x}_k, \mathbf{y}_k) \cdot \dot{L}_{\mathbf{z}_j}(\mathbf{x})}{c_k(\mathbf{z}_j)} \cdot \overset{\text{D}}{L}_{Z_k}(\mathbf{z}_j) \end{array} \right) \tag{57}$$

where

$$\overset{\text{ND}}{L}_{Z_k} = \frac{\left(\begin{array}{c} \sum_{i=0}^{m_k} (m_k - i)! \cdot p^\kappa(m_k - i) \\ \cdot \sigma_i(Z_k) \cdot \dot{G}^{(i+1)}(\phi_k) \end{array} \right)}{\left(\begin{array}{c} \sum_{l=0}^{m_k} (m_k - l)! \cdot p^\kappa(m_k - l) \\ \cdot \sigma_l(Z_k) \cdot \dot{G}^{(l)}(\phi_k) \end{array} \right)} \tag{58}$$

and

$$\overset{\text{D}}{L}_{Z_k}(\mathbf{z}_j) = \frac{\left(\begin{array}{c} \sum_{i=0}^{m_k - 1} (m_k - i - 1)! \cdot p^\kappa(m_k - i - 1) \\ \cdot \sigma_i(Z_k - \{\mathbf{z}_j\}) \cdot \dot{G}^{(i+1)}(\phi_k) \end{array} \right)}{\left(\begin{array}{c} \sum_{l=0}^{m_k} (m_k - l)! \cdot p^\kappa(m_k - l) \\ \cdot \sigma_l(Z_k) \cdot \dot{G}^{(l)}(\phi_k) \end{array} \right)} \tag{59}$$

and where

$$\phi_k = \int (1 - \dot{p}_D(\mathbf{x}_k, \mathbf{y}_k)) \cdot \dot{s}(\mathbf{x}_k, \mathbf{y}_k) d\mathbf{x}_k d\mathbf{y}_k \tag{60}$$

$$\sigma_i(Z_k) = \sigma_{m,i}\left(\frac{\hat{\tau}_k(\mathbf{z}_1)}{c_k(\mathbf{z}_1)}, \ldots, \frac{\hat{\tau}_k(\mathbf{z}_{m_k})}{c_k(\mathbf{z}_{m_k})} \right) \tag{61}$$

and

$$\sigma_i(Z_k - \{\mathbf{z}_j\}) = \sigma_{m_k - 1, i}\left(\begin{array}{c} \frac{\hat{\tau}_k(\mathbf{z}_1)}{c_k(\mathbf{z}_1)}, \ldots, \widehat{\frac{\hat{\tau}_k(\mathbf{z}_j)}{c_{k+1}(\mathbf{z}_j)}}, \\ \ldots, \frac{\hat{\tau}_k(\mathbf{z}_{m_k})}{c_k(\mathbf{z}_{m_k})} \end{array} \right); \tag{62}$$

and where

$$\hat{\tau}_k(\mathbf{z}) = \int \dot{p}_D(\mathbf{x}_k, \mathbf{z}) \cdot \dot{s}(\mathbf{x}_k, \mathbf{z}) d\mathbf{x}_k; \tag{63}$$

and where

$$y_1, \ldots, \widehat{y_j}, \ldots, y_m$$

indicates that y_j has been removed from the list $y_1, \ldots, y_m$.

Finally, the measurement-updated cardinality distribution and p.g.f. are, respectively,

$$\dot{p}(\dot{n}_k | Z_{1:k}) = \frac{\dot{\ell}_{Z_k}(\dot{n}_k) \cdot \dot{p}(\dot{n}_k | Z_{1:k-1})}{\sum_{l \geq 0} \dot{\ell}_{Z_k}(l) \cdot \dot{p}(l | Z_{1:k-1})} \tag{64}$$

and

$$\dot{G}(\dot{x}_k) = \frac{\left(\begin{array}{c} \sum_{j=0}^{m_k} \dot{x}_k^j \cdot (m_k - j)! \cdot p^\kappa(m_k - j) \\ \cdot \dot{G}^{(j)}(\dot{x}_k \cdot \phi_k) \cdot \sigma_j(Z_k) \end{array} \right)}{\left(\begin{array}{c} \sum_{i=0}^{m_k} (m_k - i)! \cdot p^\kappa(m_k - i) \\ \cdot \dot{G}^{(i)}(\phi_k) \cdot \sigma_i(Z_k) \end{array} \right)} \tag{65}$$

where

$$\dot{\ell}_{Z_{k+1}}(\dot{n}_k) = \frac{\left(\begin{array}{c} \sum_{j=0}^{\min\{m_k, \dot{n}_k\}} (m_k - j)! \cdot p^\kappa(m_k - j) \\ \cdot j! \cdot C_{\dot{n}_k, j} \cdot \phi_k^{n-j} \cdot \sigma_j(Z_k) \end{array} \right)}{\left(\begin{array}{c} \sum_{l=0}^{m_k} (m_k - l)! \cdot p^\kappa(m_k - l) \\ \cdot \sigma_l(Z_k) \cdot \dot{G}^{(l)}(\phi_k) \end{array} \right)}. \tag{66}$$

6.3 PMC-CPHD Filter: State Estimation

State estimation for he PMC-CPHD filter is accomplished in the same manner as the "classical" CPHD filter.

7 Mathematical Derivations

From Bayes' rule we know that

$$f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{y}_{1:k-1}) = f(\mathbf{x}_k | \mathbf{y}_{1:k}) \cdot f(\mathbf{y}_k | \mathbf{y}_{1:k-1}) \tag{67}$$

and so

$$f(\mathbf{x}_k \mid \mathbf{y}_{1:k}) = \frac{f(\mathbf{x}_k, \mathbf{y}_k \mid \mathbf{y}_{1:k-1})}{f(\mathbf{y}_k \mid \mathbf{y}_{1:k-1})} \tag{68}$$

$$= \frac{f(\mathbf{x}_k, \mathbf{y}_k \mid \mathbf{y}_{1:k-1})}{\int f(\mathbf{x}'_k, \mathbf{y}_k \mid \mathbf{y}_{1:k-1}) d\mathbf{x}'_k}. \tag{69}$$

However,

$$f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{y}_{1:k-1}) = \int f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1} | \mathbf{y}_{1:k-1}) d\mathbf{x}_{k-1} \tag{70}$$

For,

$$f(\mathbf{x}_k, \mathbf{y}_k, \mathbf{y}_{1:k-1})$$
$$= \int f(\mathbf{x}_k, \mathbf{y}_k, \mathbf{x}_{1:k-1}, \mathbf{y}_{1:k-1}) d\mathbf{x}_{1:k-1} \quad (71)$$
$$= \int f(\mathbf{x}_{1:k}, \mathbf{y}_{1:k}) d\mathbf{x}_{1:k-1} \quad (72)$$
$$= \int f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{1:k-1}, \mathbf{y}_{1:k-1}) d\mathbf{x}_{1:k-1} \quad (73)$$
$$= \int f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{1:k-1}) d\mathbf{x}_{k-1} \quad (74)$$

from which Eq. (70) and thus Eq. (12) follow.

We are to prove Eq. (13). For from Eq. (70) we get

$$f(\mathbf{x}_k, \mathbf{y}_k, \mathbf{y}_{1:k-1}) = \int f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{1:k-1}) d\mathbf{x}_{k-1|k-1} \quad (75)$$
$$= \int f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}, \mathbf{y}_{1:k-2}) d\mathbf{x}_{k-1} \quad (76)$$

which results in Eq. (13):

$$f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{y}_{1:k-1})$$
$$= \frac{\left(\begin{array}{c} \int f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1}, \mathbf{y}_{1:k-2}) d\mathbf{x}_{k-1} \end{array} \right)}{\int f(\mathbf{x}'_{k-1}, \mathbf{y}_{k-1}, \mathbf{y}_{1:k-2}) d\mathbf{x}'_{k-1}} \quad (77)$$
$$= \frac{\left(\begin{array}{c} \int f(\mathbf{x}_k, \mathbf{y}_k | \mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \\ \cdot f(\mathbf{x}_{k-1}, \mathbf{y}_{k-1} | \mathbf{y}_{1:k-2}) d\mathbf{x}_{k-1} \end{array} \right)}{\int f(\mathbf{x}'_{k-1}, \mathbf{y}_{k-1} | \mathbf{y}_{1:k-2}) d\mathbf{x}'_{k-1}}. \quad (78)$$

8 Conclusion

The recent conference paper [2] initiated a study of random finite set (RFS) filters for multitarget applications that are subject to relaxed independence assumptions. The main object of study is the *multitarget pairwise Markov chain* (M-PMC) model described in Section 3. The M-PMC model addresses situations in which the current multitarget state-set can depend on the previous multitarget measurement-set, and in which the current multitarget measurement-set can depend the previous multitarget measurement-set and on both the current and previous multitarget state-sets:

$$f(X_k | X_{k-1}, Y_{k-1}) \neq f(X_k | Y_{k-1}), \quad (79)$$
$$f(Y_k | X_k, X_{k-1}, Y_{k-1}) \neq f(Y_k | X_k). \quad (80)$$

This paper addressed a simpler problem: multitarget hidden Markov chain (M-HMC) models for targets that have PMC dynamics. A theoretical foundation for such models was developed. It was shown that this foundation allows one to construct PMC analogs of any RFS filter that is based on the standard multitarget motion and measurement models. In particular, it was shown that the Petetin-Desbouvries PMC-PHD filter [5] arises as a special case. As an illustration, a PMC-CPHD filter generalization of that filter was described. Thus it is now possible to track multiple targets whose dynamics are not necessary govered by restrictive HMC independence assumptions. Petetin and Desbouvries have given concrete application and implementation examples of how this can be accomplished [5].

It should be emphasized that the line of investigation in [2] has merely initiated the study of M-PMC systems. Many questions remain. For example, the definition of the M-PMC transition function $f(X_k,Y_k|X_{k-1},Y_{k-1})$ in [2] was chosen in part because it leads to potentially tractable formulas. A more intuitive understanding of physically reasonable M-PMC transition functions is required.

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