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Detectability Analysis of Detection and Estimation of Structured Action from Cluttered Data

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***Abstract*—There is good reason to model an asymmetric threat (a structured action such as a terrorist attack) as an HMM. Thence there is a means (described in earlier work) to detect it via the novel Bernoulli filter paradigm that is emerging as an integrated tracker/track-management tool. This paper details additional progress made to model the detectability of a hidden Markov model (HMM) that is observed in the presence of false measurements or clutter. The ultimate goal of this analysis is to be able to make statements regarding the minimum complexity that an HMM would need to involve in order that it be detectable with reasonable fidelity, as well as upper bounds on the level of clutter (expected number of false measurements) and probability of miss of a relevant observation. Put simply, if a threat modeled as an HMM has (say) three components – transaction O1, followed by O2 and then O3 with modeled delays in between – then this would only be detectable if the delays were very small or if there were very little clutter. A more feasible situation would involve 20 or 30 transactions. To characterize this more fully is the goal of this manuscript.**

I. Introduction

The term *asymmetric threat* refers to tactics employed by, e.g., terrorist groups to carry out attacks on a superior opponent, while trying to avoid direct confrontation. Terrorist groups are elusive, secretive, amorphously structured decentralized entities that often appear unconnected. Analysis of prior terrorist attacks suggests that a high magnitude terrorist attack requires certain enabling events to take place.

In this paper terrorist activites are modeled using Hidden Markov Models (HMMs). In previous work HMMs have been shown to provide powerful statistical techniques, and they have been applied to various problems such as speech recognition, DNA sequence analysis, robot control, fault diagnosis, and signal detection, to name a few. Excellent tutorials on HMMs can be found in [6], [7]. The applicability of HMMs for terrorist activity modeling and other national security problem situations has been illustrated in previous work, see e.g. [3], [8], [10]–[12], [14]. For example, [3] uses HMMs to identify groups with suspicious behaviour, and [8] uses HMMs for pattern recognition of international crises.

A number of different terrorist plan HMMs are proposed in [10]–[12], [14], including models for a truck bombing, a plane hijacking, and production of weapons grade material. The truck bombing HMM is shown in Figure 1. These HMMs include multiple paths from plan initiation to plan completion, following the intuition that there are multiple ways to, e.g., hijack a plane. An empirical HMM can be constructed using available prior information, or with the help from experienced intelligence analysts [11]. For example, the HMM for *development of a nuclear weapons program* (DNWP) in [10] is gleaned using the open sources [2], [5], [9], [13], [15].

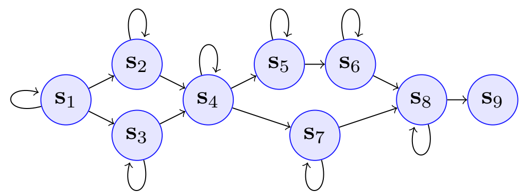

Fig. 1. Markov chain network modeling the planning of a truck bombing, taken from [11]. $\mathbf{s}_1$: Selection of targets and reconnaissance. $\mathbf{s}_2$: Set up A_1 cell. $\mathbf{s}_3$: Set up A cell. $\mathbf{s}_4$: Acquire money for operation. $\mathbf{s}_5$: Gather resources. $\mathbf{s}_6$: Expert arrives to assemble bombs. $\mathbf{s}_7$: Target reconnaissance. $\mathbf{s}_8$: Communications and final setup. $\mathbf{s}_9$: Attack.

The basic motivation for modeling terrorist activities via HMMs is twofold. Firstly, carrying out a terrorist activity requires planning and preparations, following steps that form a pattern. This pattern of actions can be modeled using a Markov chain. Secondly, the terrorists leave detectable clues about these enabling events in the observation space. The clues are not direct observations of the planning and preparations, but are rather related to them, meaning that the states in the Markov model are hidden. For example, an observation of a purchase of chemicals could be indicative of intentions to produce a chemical weapon. However, a purchase of chemicals could very well be a benign event, which motivates inclusion of a model of observations that are unrelated to the HMM. Following the target tracking literature, see e.g. [1], such observations are here designated as clutter observations.

A Bernoulli filter approach to HMM detection and estimation is presented in [4]. The Bernoulli filter can process a sequence of observations and detect if there is an activity being planned and organized, and if so, what stage of planning the activity is in. In this paper we present a detectability analysis of the problem. The problem is cast as a detection problem, where the hypotheses HMM and NO-HMM are compared using a log-likelihood ratio test. The type I and type II error probabilites are evaluated by means of model simplifications and approximations. The merits of the approximated error probabilities are shown in a simulation study.

II. PROBLEM FORMULATION

Let $\mathbf{Z}^k = \{\mathbf{Z}_j\}_{j=0}^k$ be a stream of observations $\mathbf{Z}_j$ from time step t_0 up to time step t_k. We consider the problem of testing two hypotheses against each other. The first (null) hypothesis and the second (alternative) hypothesis are

- H_0: The observations were generated by a clutter process.
- H_1: The observations were generated by an HMM-in-clutter process.

The log-likelihood ratio (LLR) is denoted

$$\ell(\mathbf{Z}^k) = \sum_{j=1}^{k} \ell(\mathbf{Z}_j) = \sum_{j=1}^{k} (\ell_1(\mathbf{Z}_j) - \ell_0(\mathbf{Z}_j)) \tag{1}$$

where $\ell_j(\cdot)$ is the log-likelihood function under hypothesis j. Let $L_j(\cdot) = \exp(\ell_j(\cdot))$ denote the likelihood of hypothesis j. Using an LLR threshold τ we can decide whether or not it is likely that an HMM exists,

$$\begin{cases} \ell(\mathbf{Z}^k) > \tau & : \text{ Decide } H_1 \\ \ell(\mathbf{Z}^k) < \tau & : \text{ Decide } H_0 \end{cases} \tag{2}$$

We are interested in the type I and type II error (false alarm and miss, respectively) probabilities

$$\mathsf{P}(\ell(\mathbf{Z}^k) > \tau | H_0 \text{ true}), \qquad \mathsf{P}(\ell(\mathbf{Z}^k) < \tau | H_1 \text{ true}) \tag{3}$$

and to obtain them we need the probability density functions (pdfs) $p(\ell(\mathbf{Z}^k)|H_0 \text{ true})$ and $p(\ell(\mathbf{Z}^k)|H_1 \text{ true})$.

The topic of this paper is modeling of the type II error probability. Expressing the pdf $p(\ell(\mathbf{Z}^k)|H_1 \text{ true})$ analytically is prohibitively difficult and complex in the general case. We will therefore make some simplifications and approximations such that an approximate pdf can be obtained. We will show how the mean and standard deviation of the likelihood ratio under hypothesis H_1 can be approximated such that the pdf can be approximated by a Gaussian density.

III. ASYMMETRIC THREAT MODELING

A. HMM *state*

Let $\mathbf{s}_k \in \mathcal{S}$ denote the HMM state at time t_k, where $\mathcal{S}$ is a discrete state space with N_s states, $\mathcal{S} = \{S_1, S_2, \ldots, S_{N_s}\}$. Further, let $\mathbf{t}_k \in \mathcal{T} = \{0, 1\}$ denote the transition state, defined as $\mathbf{t}_k = 1$ if $\mathbf{s}_k \neq \mathbf{s}_{k-1}$ and $\mathbf{t}_k = 0$ otherwise. The state transitions are important because in the variant of HMMs used here the observations become available only upon state transitions. The auxiliary transition variable $\mathbf{t}$ is used because the authors found that it simplifies mathematical analysis and implementation. Let $\zeta_k = (\mathbf{s}_k, \mathbf{t}_k)$ denote the joint variable.

B. *State transitions*

For the joint transition probability $\pi(\zeta_k|\zeta_{k-1}) = \pi(\mathbf{s}_k, \mathbf{t}_k|\mathbf{s}_{k-1}, \mathbf{t}_{k-1})$ the following holds

$$\pi(\zeta_k|\zeta_{k-1}) = \pi(\mathbf{t}_k|\mathbf{s}_k, \mathbf{s}_{k-1}, \mathbf{t}_{k-1})\pi(\mathbf{s}_k|\mathbf{s}_{k-1}). \tag{4}$$

The HMM state transitions follow a first order Markov chain with transition probability $\pi(\mathbf{s}_k|\mathbf{s}_{k-1})$. For the transition state $\mathbf{t}_k$ the transition matrix is

$$\Pi = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ if } \mathbf{s}_k \neq \mathbf{s}_{k-1}; \quad \Pi = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ otherwise.} \tag{5}$$

C. *Observations*

The observations $\mathbf{z}_k \in \mathcal{Z}$ are discrete random variables, where $\mathcal{Z}$ is a discrete state space with N_z states, $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_{N_z}\}$. With a state dependent probability of detection

$$p_{\mathrm{D}}(\zeta_k) = \begin{cases} p_{\mathrm{D}}^0 \in (0, 1) & \text{if } \mathbf{t}_k = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

the HMM generates an observation $\mathbf{z}_k$. The observation process is defined by the likelihood

$$h(\mathbf{z}_k|\zeta_k) = g_s(\mathbf{z}_k|\mathbf{s}_k) \tag{7}$$

There are also clutter observations (false alarms) superimposed on the true HMM observations. In each time-step, with probability $0 < p_{\mathrm{FA}} < 1$ a clutter observation is generated as a random sample from a process with probability mass function (pmf) $g_{\mathrm{FA}}(\mathbf{z}_k)$.

Let $\mathbf{Z}_k$ be the random finite set (RFS) observation at time t_k. Let $\mathbf{Z}^k$ denote all such observations from time t_1 to t_k, $\mathbf{Z}^k = \{\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_k\}$. If an HMM does not exists then $\mathbf{Z}_k = \mathbf{C}_k$ and if an HMM exists then $\mathbf{Z}_k$ is the union of two independent RFS,

$$\mathbf{Z}_k = \mathbf{W}_k \cup \mathbf{C}_k, \tag{8}$$

where $\mathbf{W}_k$ is HMM generated observations and $\mathbf{C}_k$ is clutter observations.

The clutter observations are modeled as a Bernoulli RFS with Finite Set Statistics (FISST) pdf

$$\kappa(\mathbf{Z}) = \begin{cases} 1 - p_{\mathrm{FA}}, & \text{if } \mathbf{Z} = \emptyset \\ p_{\mathrm{FA}} g_{\mathrm{FA}}(\mathbf{z}), & \text{if } \mathbf{Z} = \{\mathbf{z}\} \end{cases} \tag{9}$$

The HMM observations are modeled as a Bernoulli RFS with FISST pdf

$$\eta(\mathbf{Z}|\zeta) = \begin{cases} 1 - p_{\mathrm{D}}(\zeta), & \text{if } \mathbf{Z} = \emptyset \\ p_{\mathrm{D}}(\zeta) g_s(\mathbf{z}|\zeta), & \text{if } \mathbf{Z} = \{\mathbf{z}\} \end{cases} \tag{10}$$

The prior and posterior HMM state pmfs are denoted $P_{k|k-1}(\zeta)$ and $P_{k|k}(\zeta)$. The observation pdf is

$$\varphi(\mathbf{Z}|\zeta) = \sum_{\mathbf{W} \subseteq \mathbf{Z}} \eta(\mathbf{W}|\zeta)\kappa(\mathbf{Z} \backslash \mathbf{W}), \tag{11}$$

where $\backslash$ denotes set difference. The summation then has two different cases

$$\varphi(\mathbf{Z}|\zeta) = \begin{cases} \eta(\emptyset|\zeta)\kappa(\emptyset) & \text{if } \mathbf{Z} = \emptyset, \\ \eta(\mathbf{z}|\zeta)\kappa(\emptyset) + \eta(\emptyset|\zeta)\kappa(\mathbf{z}) & \text{if } \mathbf{Z} = \{\mathbf{z}\}. \end{cases} \tag{12}$$

D. *Likelihood ratio*

For H_0 the likelihood is

$$L_0(\mathbf{Z}_j) = \kappa(\mathbf{Z}_j) \tag{13}$$

and for H_1 the likelihood is

$$L_1(\mathbf{Z}_j) = \int \varphi(\mathbf{Z}_j|\zeta) P_{j|j-1}(\zeta) \mathrm{d}\zeta \tag{14}$$

The likelihood ratio $L(\mathbf{Z}_j) = L_1(\mathbf{Z}_j)/L_0(\mathbf{Z}_j)$ has two cases,

$$L_j^{\emptyset} = 1 - p_{\mathrm{D}}^0 P_{j|j-1}(\mathbf{t}=1) \tag{15}$$

$$L_j^{\mathbf{z}} = 1 - p_{\mathrm{D}}^0 P_{j|j-1}(\mathbf{t}=1) + \frac{1-p_{\mathrm{FA}}}{p_{\mathrm{FA}} g_{\mathrm{FA}}(\mathbf{z})} \times p_{\mathrm{D}}^0 \int h(\mathbf{z}|\mathbf{s},\mathbf{t}=1) P_{j|j-1}(\mathbf{s},\mathbf{t}=1) \mathrm{d}\mathbf{s} \tag{16}$$

where $L_j^{\emptyset} = L(\mathbf{Z}_j = \emptyset)$ and $L_j^{\mathbf{z}} = L(\mathbf{Z}_j = \{\mathbf{z}\})$.

IV. Simplifications and approximations

A. Model simplifications

1) The HMM is in the form of a "daisy-chain", i.e. the HMM state s can only transition to the next state or remain in the same state. Expressed in terms of the transition probability $\pi(\mathbf{s}_k|\mathbf{s}_{k-1})$, if $\mathbf{s}_k = S_i$, $1 \leq i < N_s$ then

$$\pi(\mathbf{s}_k = S_j|\mathbf{s}_{k-1} = S_i) = \begin{cases} 1 - P_T & j = i \\ P_T & j = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

and if $\mathbf{s}_k = S_{N_s}$ then

$$\pi(\mathbf{s}_k = S_j|\mathbf{s}_{k-1} = S_i) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

2) The size of the observation state space is equal to the HMM state space, i.e. $N_z = N_s$. The observation pmf is

$$g_s(\mathbf{z} = Z_j|\mathbf{s}_k = S_i) = \begin{cases} P_{obs} & \text{if } j = i \\ \frac{1-P_{obs}}{N_z - 1} & \text{otherwise} \end{cases} \tag{19}$$

where $0 \ll P_{obs} \lesssim 1$ (i.e. P_{obs} is close to one).
Remark: This simplification means that each state has its own unique type of detection, and it is very unlikely that – given that there is a detection – a state would give the "wrong" type of detection. For example, let there be two states representing 1) that an apartment has been rented, and 2) that a large quantity of fertilizer has been bought. Given that the state is detected, we assume that it is unlikely that apartment rental will produce a true detection that fertilizer was bought, or vice versa. However, note that we do not make any assumptions regarding the probability that a state is detected. □

3) The clutter is uniformly distributed, $g_{\mathrm{FA}}(\mathbf{z}) = {N_z}^{-1}$, i.e. it is equiprobable for all the N_z possibilities.

B. Likelihood approximations

1) Under the assumption that the HMM is a simple chain, the predicted marginal probability of state transition is

$$P_{j|j-1}(\mathbf{t}=1) = P_T \left(1 - P_{j|j-1}(\mathbf{s} = S_{N_s})\right) \tag{20}$$

We approximate this as

$$P_{j|j-1}(\mathbf{t}=1) \approx P_T. \tag{21}$$

i.e. the approximation is $P_{j|j-1}(\mathbf{s} = S_{N_s}) \approx 0$. Under this approximation it follows that the likelihood ratio at time steps for which there is no detection (15) is approximated as

$$L_j^{\emptyset} \approx 1 - p_{\mathrm{D}}^0 P_T \tag{22}$$

Remark: The approximation $P_{j|j-1}(\mathbf{s} = S_{N_s}) \approx 0$ is typically valid when the true state is not close to the last state S_{N_s}. In general the approximation is more accurate the more states the HMM has, i.e. the larger N_s is. □

2) Assume that at time step t_m there is a detection $\mathbf{z}_m = Z_i$. For the observation likelihood function (19) the integral in the likelihood ratio (16) is

$$\int h(\mathbf{z} = Z_i|\mathbf{s},\mathbf{t}=1) P_{m|m-1}(\mathbf{s},\mathbf{t}=1)\mathrm{d}\mathbf{s} = P_{obs} P_{m|m-1}(\mathbf{s} = S_i, \mathbf{t}=1) + \frac{1-P_{obs}}{N_z - 1}\left(1 - P_{m|m-1}(\mathbf{s} = S_i,\mathbf{t}=1)\right) \tag{23}$$

We approximate

$$\frac{1-P_{obs}}{N_z - 1} \approx 0 \tag{24}$$

and it follows that the likelihood ratio is approximated

$$L_m^{\mathbf{z}} \approx 1 - p_{\mathrm{D}}^0 P_T + \frac{1-p_{\mathrm{FA}}}{p_{\mathrm{FA}} N_s^{-1}} p_{\mathrm{D}}^0 P_{obs} P_{m|m-1}(\mathbf{s} = S_i,\mathbf{t}=1) \tag{25}$$

Remark: The approximation used here is valid under the model simplification above that P_{obs} is almost one. Further, in general the approximation is more accurate the larger $N_z = N_s$ is, i.e. the more states there are in the HMM. □

V. Approximation of the type II error probability

When H_1 is true, an HMM exists and the detections $\mathbf{Z}^k$ are generated by both the HMM and the clutter process. For either type of detection, the likelihood ratio can be further simplified.

1) If the detection is a clutter detection, the typical case is that $\mathbf{s}_m = S_i$ has low predicted probability ($P_{m|m-1}(\mathbf{s} = S_i, \mathbf{t} = 1) \approx 0$). In this case the likelihood ratio (25) can be further approximated as

$$L_m^{\mathbf{z}} \approx 1 - p_{\mathrm{D}}^0 P_T \tag{26}$$

Remark: The approximation $P_{m|m-1}(\mathbf{s} = S_i,\mathbf{t}=1) \approx 0$ is more accurate the more states there are, i.e. the larger N_s is. Note that $(1-p_{\mathrm{FA}})/p_{\mathrm{FA}}$ in (25) becomes increases for decreasing p_{FA}, and this (together with an increaseing number of states N_s) will make the approximation less accurate. However, a very low p_{FA} is of little practical interest, because in most realistic scenarios the probability of have a clutter detection will not be close to zero. □

2) If the detection $\mathbf{z}_m = Z_i$ was caused by the HMM, the approximation $P_{m|m-1}(\mathbf{s}_k = S_i, \mathbf{t}_k = 1) \approx 0$ typically does not hold. For this case the predicted probability is simplified as follows.

Assume that at time step $n < m$ there was a detection $\mathbf{z}_n = Z_{i-1}$ such that the posterior pmf $P_{n|n}(\zeta)$ indicates a high probability that the HMM is in state $\mathbf{s}_n = S_{i-1}$. Further, assume that in between time steps n and m there were no detections. Given the pmf $P_{n|n}(\mathbf{s}_n, \mathbf{t}_n)$ the sought after predicted probability $P_{m|m-1}(\mathbf{s}_m = S_i, \mathbf{t}_m = 1)$ is then given by first iterating prediction and no-detection-measurement update for $N = m - n$ time steps, and then evaluating for $\mathbf{s} = S_i$ and $\mathbf{t} = 1$. By approximating $P_{n|n}(\cdot)$ as follows,

$$P_{n|n}(\mathbf{s} = S_s, \mathbf{t} = t) \approx \begin{cases} 0.9 & \text{if } s = i-1, \quad t = 1 \\ 0.09 & \text{if } s = i-2, \quad t = 0 \\ 0.01 & \text{if } s = i-3, \quad t = 0 \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

the probability $P_{m|m-1}(\mathbf{s}_m = S_i, \mathbf{t}_m = 1)$ can be approximately expressed as a function of the number of time steps N since the last measurement update. We denote this probability as $\bar{P}(N)$. Thus, the likelihood ratio (25) is approximated by

$$L_m^{\mathbf{z}} \approx 1 - p_{\mathrm{D}}^0 P_T + \frac{1 - p_{\mathrm{FA}}}{p_{\mathrm{FA}} N_s^{-1}} p_{\mathrm{D}}^0 P_{obs} \bar{P}(N) \tag{28}$$

Remark: The specific numerical values in (27) are motivated as follows: the majority of the probability mass is concentrated in the same state as the detection indicates. Some probability mass is contained in the two previous states — a reflection of the probability that the detection was a false alarm and the state $\mathbf{s}$ has not transitioned to S_{i-1} after all. Empirically we have found that these values are accurate for $p_{\mathrm{FA}} > 10\%$. For $p_{\mathrm{FA}} \approx 1\%$ almost all probability is concentrated in the state S_{i-1}, however such low p_{FA} are of little to no practical interest. Further, empirically we have found that the values are accurate for all p_{D}^0. □

Now, let N_t be the total number of time steps that it takes for the HMM to transition from the first to the last state. The HMM has to pass through each state, meaning that $N_t \geq N_s$. The number of time steps the HMM state will remain in a specific state S_i (i.e. no state transition) is well known to be a random variable that is geometrically distributed with parameter P_T. The total number of time steps the HMM state will remain in the same state (i.e. no state transition) is therefore the sum of $N_s - 1$ geometrically distributed random variables, each with parameter P_T. A sum of $N_s - 1$ identically geometrically distributed random variables is well known to be negative binomial distributed with parameters $N_s - 1$ and P_T. Thus, for N_t we have the following pmf

$$P(N_t) = \begin{cases} 0 & N_t < N_s \\ \mathcal{NBIN}(N_t - N_s; N_s - 1, P_T) & \text{otherwise} \end{cases} \tag{29}$$

where

$$\mathcal{NBIN}(k; r, p) = \binom{k + r - 1}{k} (1 - p)^r p^k \tag{30}$$

Given N_t, let $N_d \in \{0, \ldots, N_t\}$ be the number of "true" detections (if the state just transitioned to S_i, the "true" detection is Z_i). The probability of a true detection is $p_{\mathrm{D}}^0 P_{obs}$. The detections are assumed to be independent of each other, and the conditional pmf for N_d is binomial distributed

$$\begin{aligned} P(N_d|N_t) &= \mathcal{BIN}(N_d; N_t, p_{\mathrm{D}}^0 P_{obs}) && (31) \\ &= \binom{N_t}{N_d} \left(p_{\mathrm{D}}^0 P_{obs}\right)^{N_d} \left(1 - p_{\mathrm{D}}^0 P_{obs}\right)^{N_t - N_d} && (32) \end{aligned}$$

For a given N_t and N_d we assume that the N_d detections are uniformly distributed over the N_t time steps, i.e. there are N_t/N_d time steps between each of the "true" detections. Under this assumption, and using the approximations above, the LLR ℓ is approximated by

$$\begin{aligned} \hat{\ell}(N_t, N_d) = {} & (N_t - N_d) \log(1 - p_{\mathrm{D}}^0 P_T) && (33) \\ & + N_d \log(1 - p_D^0 P_T + \frac{1 - p_{\mathrm{FA}}}{p_{\mathrm{FA}} N_s^{-1}} P_{obs} \bar{P}(N_t/N_d)) \end{aligned}$$

For a given measurement sequence $\mathbf{Z}^k$, the resulting LLR $\ell(\mathbf{Z}^k)$ is deterministic. The probability density of ℓ, conditioned on N_t and N_d is approximated as

$$p(\ell | N_t, N_d) \approx \delta(\ell = \hat{\ell}(N_t, N_d)) \tag{34}$$

where $\delta(\cdot)$ is the Dirac delta function. The cth moment of the LLR is approximated as follows

$$\begin{aligned} \mathbb{E}\left[\ell^c | H_1 \text{ true}\right] &= \int \ell^c p(\ell) \mathrm{d}\ell && (35) \\ &\approx \int \sum_{N_t} \sum_{N_d} \ell^c p(\ell | N_t, N_d) P(N_d|N_t) P(N_t) \mathrm{d}\ell && (36) \\ &\approx \sum_{N_t} \sum_{N_d} \left(\hat{\ell}(N_t, N_d)\right)^c P(N_d|N_t) P(N_t) && (37) \end{aligned}$$

We approximate the true pdf over ℓ with a Gaussian pdf

$$p(\ell | H_1 \text{ true}) \approx \mathcal{N}\left(\ell\,;\, \hat{\mu}_\ell^1, \hat{\sigma}_\ell^1\right) \tag{38}$$

where the mean and standard deviation are given as

$$\begin{aligned} \hat{\mu}_\ell^1 &= \mathbb{E}\left[\ell | H_1 \text{ true}\right] && (39) \\ \hat{\sigma}_\ell^1 &= \mathbb{E}\left[\ell^2 | H_1 \text{ true}\right] - \mathbb{E}\left[\ell | H_1 \text{ true}\right]^2 && (40) \end{aligned}$$

VI. Results

We generated HMMs with parameters

$$\begin{aligned} & N_s \in \{10, 20, 30, 40\} && P_T \in \{0.05, 0.1, 0.30\} \\ & p_{\mathrm{D}}^0 \in \{0.7, 0.8, 0.9\} && p_{\mathrm{FA}} \in \{0.1, 0.2, 0.3\} \qquad (41) \\ & P_{obs} = 0.99 \end{aligned}$$

which gives 108 different HMMs. Each HMM was simulated 1000 times, and we compared the empirical distribution for the likelihood L with our approximation. For brevity we do not show all the results, instead selected results are shown in Figures 2, 3, 4, and 5. We see that the proposed Gaussian approximation is least accurate when both N_s and P_T are low. For higher N_s the approximation is quite accurate for all P_T, p_{D}^0 and p_{FA}.

VII. Conclusion

In previous work the "Adaptive Safety and Monitoring" (ASAM) framework was introduced as a means to model asymmetric threats such as terrorist attacks. The modeling recognizes both the sequential nature of the activity – for example, a planning step must precede a surveillance step which often is followed by (but may be in parallel to) a funding step – and also incorporates statistical uncertainty. The natural framework is a hidden Markov model (HMM).

More recently, the data association aspects to the model have been examined: the steps above may be observed or they may be missed; and even if observed they will surely be awash in irrelevant "clutter" observations. As such, more recent work has focused on applying insight from target tracking theory where such data association problems are commonplace. A natural algorithm has arisen – the Bernoulli filter – and it seems to work well.

But a natural question remains: can these sorts of activities be detected at all? On an intuitive level, for detectability there must be maximum levels of clutter allowable; maximum intervals between relevant observations; and a minimum level of complexity. In this paper we have attempted to address the issue by approximating the detectability of such a process. Much remains: the parallel "false alarm" process must also be analyzed in order to make defensible detectability statements for particular parameterizations. The goal, which with that missing piece should be attainable, is to make "back of the envelope" predictions, and thereby, ultimately, to suggest collection strategies.

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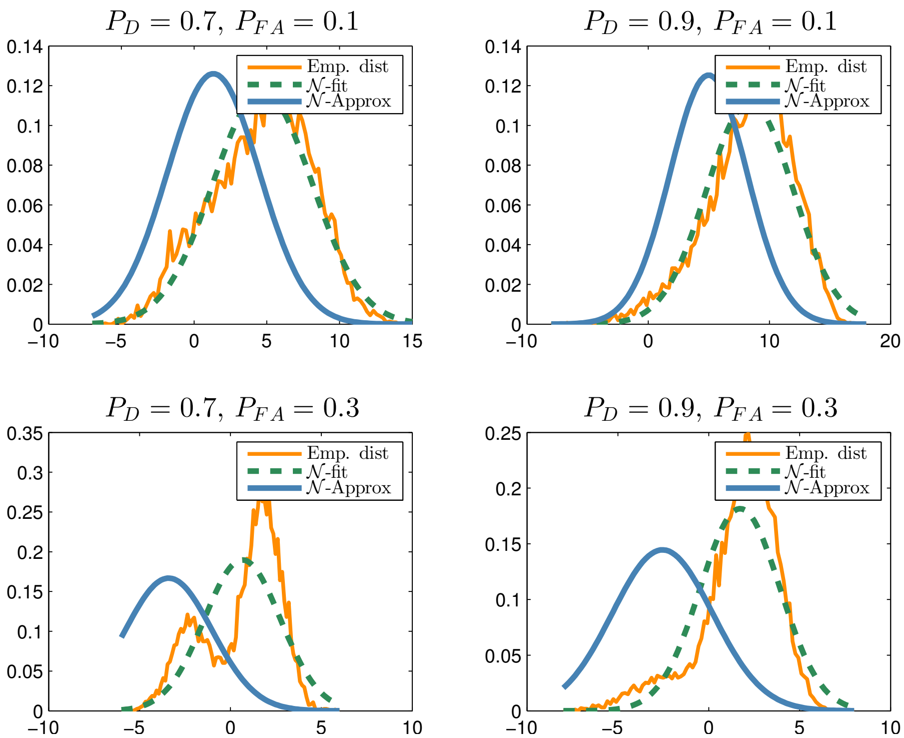

Fig. 2. An HMM exists, results for $N_s = 10$ and $P_T = 0.05$. Empirical distribution (solid orange) compared to Gaussian fitted to the data (dotted green) and Gaussian approximation (dashed blue).

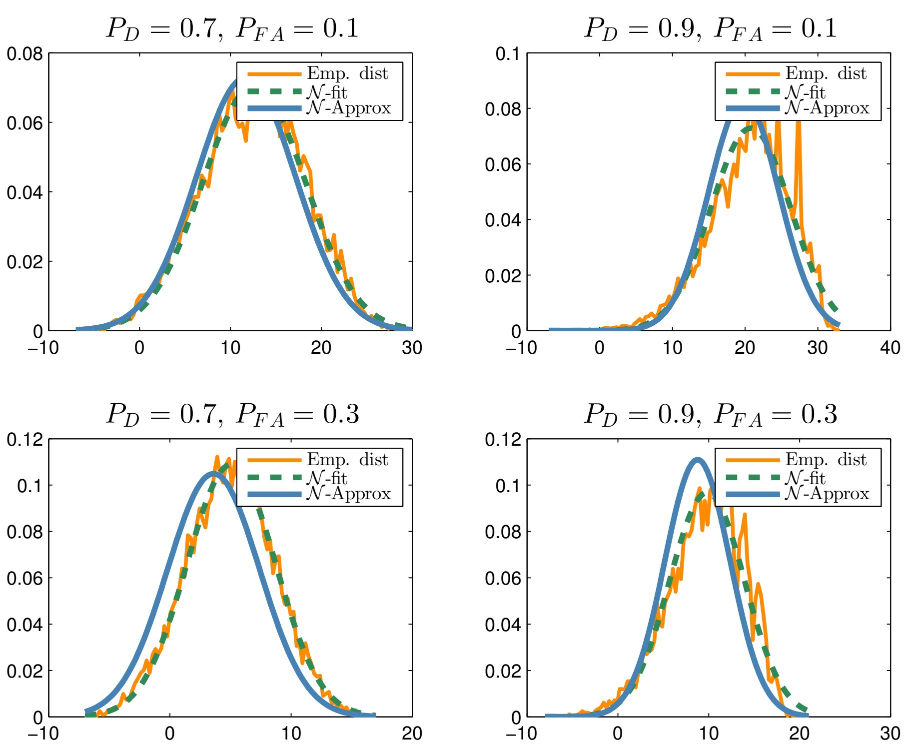

Fig. 3. An HMM exists, results for $N_s = 10$ and $P_T = 0.30$. Empirical distribution (solid orange) compared to Gaussian fitted to the data (dotted green) and Gaussian approximation (dashed blue).

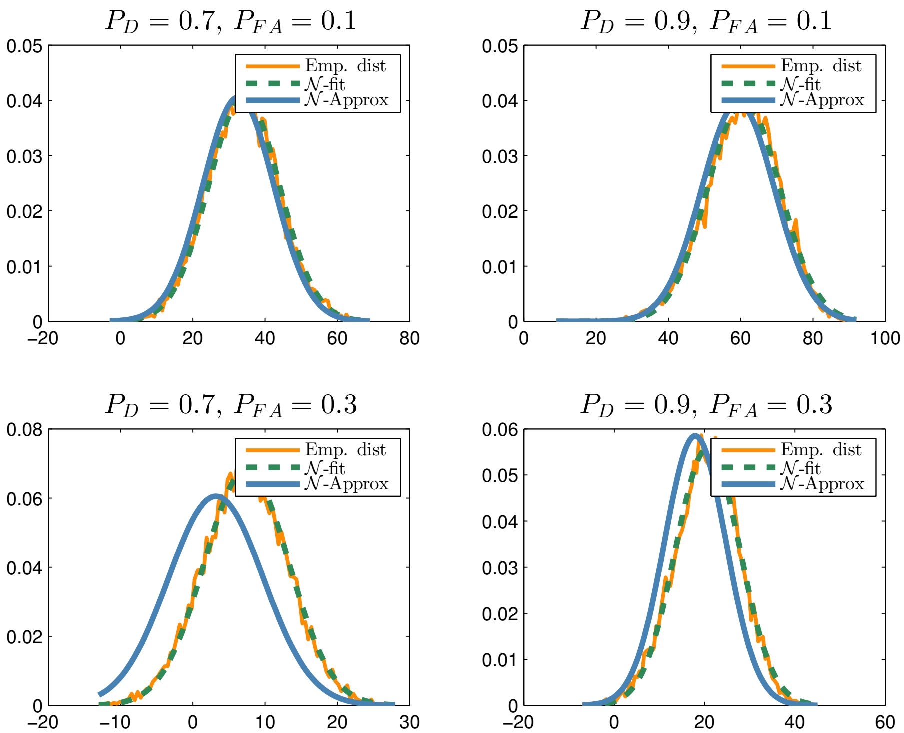

Fig. 4. An HMM exists, results for $N_s = 40$ and $P_T = 0.05$. Empirical distribution (solid orange) compared to Gaussian fitted to the data (dotted green) and Gaussian approximation (dashed blue).

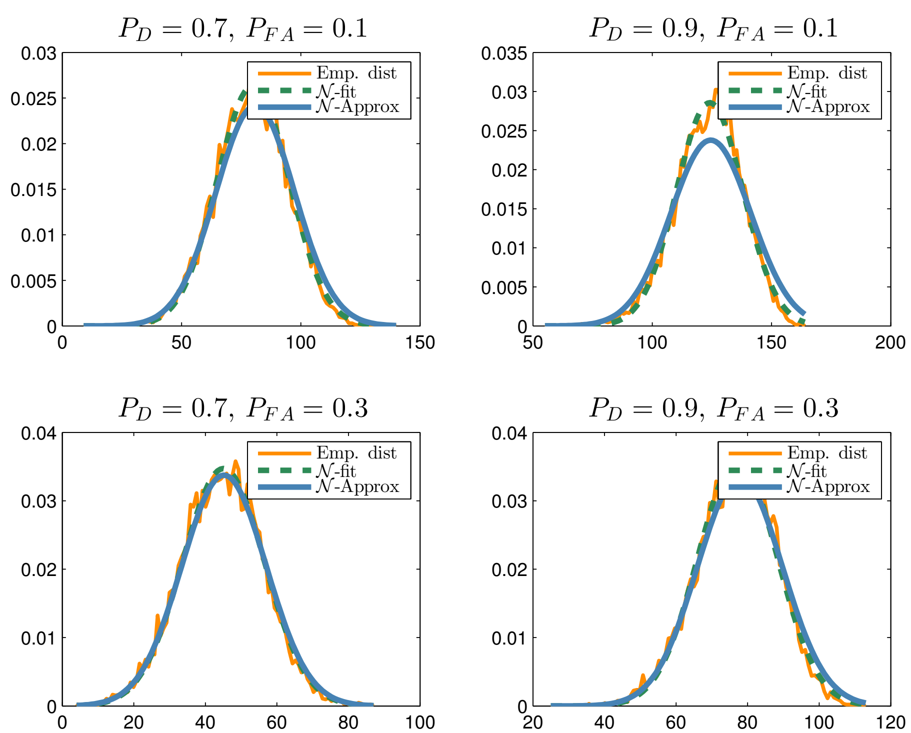

Fig. 5. An HMM exists, results for $N_s = 40$ and $P_T = 0.30$. Empirical distribution (solid orange) compared to Gaussian fitted to the data (dotted green) and Gaussian approximation (dashed blue).